

Arnold Diffusion in a priori chaotic Hamiltonian systems

V.Gelfreich* and D.Turaev***

* Mathematics Institute, University of Warwick
v.gelfreich@warwick.ac.uk

** Department of Mathematics, Imperial College, London
d.turaev@imperial.ac.uk

December 2, 2014

Abstract

We assume that a symplectic real-analytic map has an invariant normally hyperbolic cylinder and an associated transverse homoclinic cylinder. It is well known that such cylinder is preserved under small perturbations. We prove that for a generic real-analytic perturbation of this map the boundaries of the cylinder are connected by trajectories of the perturbed map.

*This work was supported by the grant RSF 14-41-00044.

1 Introduction

A Hamiltonian dynamical system is defined with the help of a Hamilton function $H : M \rightarrow \mathbb{R}$ on a symplectic manifold M of dimension $2n$. Let M_c be a connected component of a level set $\{H = c\}$. Since the Hamiltonian remains constant along the trajectories of the Hamiltonian system, the set M_c is invariant. Depending on the Hamilton function H and the energy c , the restriction of the dynamics onto M_c may vary from uniformly hyperbolic (e.g., in the case of a geodesic flow on a surface of negative curvature) to completely integrable.

Since Poincaré's works it is accepted that a typical Hamiltonian systems does not have any additional integral of motion independent of H (unless the system possesses some symmetries and Noether theorem applies). On the other hand a generic Hamiltonian system is nearly integrable in a neighbourhood of a totally elliptic equilibrium (a generic minimum or maximum of H) or totally elliptic periodic orbit. Therefore, by the Kolmogorov-Arnold-Moser (KAM) theory, one should not expect from a given Hamiltonian system to be ergodic (with respect to the Liouville measure) on every energy level [69]. Indeed, the KAM-theory establishes that under sufficiently mild conditions a nearly integrable system possesses a set of invariant tori which has positive measure.

Recall that each of the KAM tori has dimension n and, for $n > 2$, does not divide M_c which has dimension $(2n - 1)$. Moreover, for $n > 2$ the complement to the union of all KAM tori is connected and dense in M_c . So the KAM theory does not contradict to the existence of a dense orbit in M_c . It is unknown whether such orbits really exist in nearly integrable systems. The question goes back to Fermi [34] who suggested the following notion: a Hamiltonian system is called *quasi-ergodic* if in every M_c any two open sets are connected by a trajectory. This property is equivalent to topological transitivity of the Hamiltonian flow on M_c . This property can also be restated in different terms: (a) in every M_c there is a dense orbit or (b) in every M_c dense orbits form a residual subset.

Fermi conjectured [34] that quasi-ergodicity is a generic property of Hamiltonian systems, but proved a weaker statement only: if a Hamiltonian system with $n > 2$ degrees of freedom has the form

$$H = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon), \tag{1}$$

where H_0 is integrable and (I, φ) are action-angle variables, then generically M_c does not contain an invariant $(2n - 2)$ -dimensional hyper-surface which is analytic in ε . The existence of such surface would, obviously, prevent the quasi-ergodicity. However, it is not known whether a non-analytic invariant hyper-surface can exist generically, so Fermi's quasi-ergodic hypothesis remains unproved. The recent papers [71, 59, 18, 60, 68] make an important step in understanding of the underlying dynamics by showing that in a generic (in a certain smooth category) case with $2\frac{1}{2}$ or more degrees of freedom, there are trajectories which visit an a-priori prescribed sequence of balls. A paper [58] provides examples of an orbit dense in a set of maximal Hausdorff dimension (equal to 5 as $n = 3$).

This problem is closely related to the problem of stability of a totally elliptic fixed point of a symplectic diffeomorphism, or stability of a periodic totally elliptic periodic orbit for a Hamiltonian flow. It was proved in [32, 31] that stability can be broken by an arbitrarily

small smooth perturbation. It is believed that a totally elliptic periodic orbit is generically unstable but the time scales for this instability to manifest itself are extremely long.

The unperturbed system (1) is described by the Hamiltonian $H = H_0(I)$. Then the actions I are constant along trajectories, so the equation $I = I_0$ defines an invariant torus, and the angles φ are quasi-periodic functions of time with the frequency vector $\omega_0(I) = H'_0(I)$. KAM theory implies that the majority of invariant tori survive under perturbation. Tori with rationally dependent frequencies are called *resonant* and are destroyed by a typical perturbation. The frequency of a resonant torus satisfies a condition of the form $\omega_0(I) \cdot \mathbf{k} = 0$ for some $\mathbf{k} \in \mathbb{Z}^n \setminus \{0\}$. The resonant tori form a “resonant web”, typically (e.g. if ω_0 is a local diffeomorphism) a dense set of measure zero.

Arnold’s example [1] showed that a trajectory of the perturbed system (1) can drift along a resonance. This paper inspired a large number of studies in long-time stability of actions, the problem which is known as “Arnold diffusion”. It has been attracting a lot of attention recently and we refer the reader to surveys in [10, 27, 29] for a more detailed discussion.

It should be noted that the motion along the resonant web is very slow: Nekhoroshev theory [75] provides a lower bound on the instability times. Let $\{\cdot, \cdot\}$ denote the Poisson brackets. Then $\dot{I} = \{H, I\} = \varepsilon\{H_1, I\}$ is of the order of ε . On the other hand, if the system satisfies assumptions of the KAM theory, $|I(t) - I(0)|$ remains small for all times and the majority of initial conditions, i.e., for the set of initial conditions of asymptotically full measure. If H satisfies assumptions of the Nekhoroshev theory, there are some exponents $a, b > 0$ such that $|I(t) - I(0)| < \varepsilon^a$ for all $|t| < \exp \varepsilon^{-b}$ and for all initial conditions. This estimate establishes an exponentially large lower bound for the times of Arnold Diffusion.

It is important to stress that the upper bound on the speed of Arnold diffusion strongly depends on the smoothness of the system. Indeed, the stability times are exponentially large in ε^{-1} for analytic systems, but only polynomial bounds can be obtained in the C^k category. In particular, the papers [71, 59, 18] study the Arnold diffusion for non-analytic Hamiltonians and therefore the bounds established by the analytical Nekhoroshev theory are likely to be violated. The problem of genericity of Arnold diffusion in analytic category remains fully open. We believe the methods proposed in our paper will help to achieve an advancement in the analytic case.

In a neighbourhood of a simple resonance, normal form theory suggests existence of a normally hyperbolic cylinder with a pendulum-like separatrix. Bernard proved the existence of normally-hyperbolic cylinders in a priori stable Hamiltonian systems the size of which is bounded from below independently of the size of the perturbation [5].

A model for this situation is usually obtained by assuming that the integrable part of the Hamiltonian already possesses a normally-hyperbolic cylinder and an associated homoclinic loop (e.g. by considering $H_0 = P(p, q) + h_0(I)$ where P is a Hamiltonian of a pendulum). A system of this type is called *a-priori unstable*. The drift of orbits along the cylinder has been actively studied in the last decade [4, 11, 16, 17, 23, 24, 27, 84, 85], including the problem of genericity of this phenomenon and instability times. It should be noted that the Arnold diffusion is much faster in this case.

In these studies, a drifting trajectory typically stays most of the time near the normally-

hyperbolic cylinder, occasionally making a trip near a homoclinic loop. The process can be described using the notion of a scattering map introduced by Delshams, de la Llave and Seara in [28]. Earlier Moeckel [72] suggested that Arnold diffusion can be modeled by random application of two area-preserving maps on a cylinder. In this way the deterministic Hamiltonian dynamics is modeled by an iterated function system, and the obstacles to a drift along the cylinder appear in the form of common invariant curves [72, 14, 74].

In our paper we depart from the near-integrable setting and study the dynamics of an arbitrary exact symplectic map in a homoclinic channel, a neighbourhood of a normally-hyperbolic two-dimensional cylinder A along with a sequence of homoclinic cylinders B at the transverse intersection of the stable and unstable manifolds of A . We conduct a rigorous reduction of the problem to the study of an iterated function system and show that the existence of a drifting trajectory (i.e. the instability of the Arnold diffusion type) is guaranteed when the exact symplectic maps of the cylinder A that constitute the iterated function system do not have a common invariant curve. The reduction scheme is in the same spirit as in [74, 45] (while the setting and proofs are different). The completely novel result is that the existence of drifting orbits is a generic phenomenon, i.e. it holds for an open and dense subset of a neighbourhood, in the space of analytic symplectic maps, of the given map with a homoclinic channel, provided the restriction of the map on the cylinder A has a twist property. All the known similar genericity results for the Arnold diffusion have been proven so far in the smooth category and use the non-analiticity of the perturbations in an essential way.

In one respect, the situation we consider is more general than in the near-integrable setting, as we do not assume the existence of a large set of KAM curves on the invariant cylinder A . On the other side, as one can extract from the example of [21], our assumption of the strong transversality of the homoclinic intersections which we need in order to define the scattering maps that form the iteration function system seems to fail for a generic analytic near-integrable system in a neighbourhood of a resonance in the a priori stable case. Therefore, our results do not admit an immediate translation to the a priori stable case.

Rather, the problem we consider here is related to the *a-priori chaotic* case. The term refers to a Hamiltonian of the form $H = H_0(p, q, I) + H_1(p, q, I, \theta)$, where p is the group of variables conjugate to the variables q , and θ is the angular variable conjugate to I . Here H_1 is assumed to be small, and H_0 has the additional integral I but is not completely integrable. In particular, we speak about the a priori chaotic situation if there exists a non-trivial hyperbolic set Λ in the $(p, q,)$ -space for each value of the integral I from some interval $[I_a, I_b]$. Then, adding the perturbation H_1 will typically cause the “diffusion” of I .

Namely, the existence of a non-trivial hyperbolic set Λ is equivalent [82] to the existence of a hyperbolic periodic orbit whose stable and unstable manifolds have a transverse intersection along a homoclinic orbit. These two orbits depend continuously on I , so we have a continuous family of periodic orbits and a corresponding continuous family of homoclinic orbits, and we assume these two families are defined for all $I \in [I_a, I_b]$. A saddle periodic orbit in the (p, q) -space corresponds to a saddle two-dimensional invariant torus in the full (p, q, I, θ) -space. We take a cross-section of the form $F(p, q, I) = 0$ to this torus; then a Poincare map is defined on

the cross-section, and the intersection of the torus with the cross-section is a saddle invariant circle of this map. The circle is given by an equation $(p, q) = (p_I, q_I) = \text{const}$. The existence of a transverse homoclinic intersection of the stable and unstable manifolds of the saddle periodic orbit in the (p, q) -space implies the existence of a transverse homoclinic intersection of the local stable manifold of the invariant circle on the cross-section with a piece of its global unstable manifold. Since the Hamiltonian H_0 is θ -independent, i.e. the problem is symmetric with respect to rotation in θ , each connected component of the homoclinic intersection is a circle $(p, q) = (p_I^h, q_I^h) = \text{const}$. The union of the invariant circles $(p, q) = (p_I, q_I)$ over all $I \in [I_a, I_b]$ forms a normally-hyperbolic cylinder A of the Poincare map; the union of the homoclinic circles $(p, q) = (p_I^h, q_I^h) = \text{const}$ over all $I \in [I_a, I_b]$ forms a homoclinic cylinder B . It is easy to check that the *strong transversality conditions* from our Main Theorem (Theorem 2.1) are fulfilled by the homoclinic channel $\{A, B\}$.

Since H_0 is θ -independent, and I is the constant of motion, the Poincare map on the invariant cylinder A has the form

$$\bar{\theta} = \theta + \omega(I), \quad \bar{I} = I.$$

The map on the cylinder satisfies the twist condition if $\omega'(I) \neq 0$ for all I under consideration. Under this condition our Theorem 2.1 implies the following result:

Let H_0 be a real analytic function, e.g. it is holomorphic in a certain complex neighbourhood (analyticity domain) Q of a bounded set in the real (p, q, I, θ) space that contains all the orbits with the initial conditions in the cylinders A and B . There exists $\varepsilon > 0$ such that for an open and dense subset of the set of real analytic functions H_1 such that $\sup_{cl(Q)} |H_1| < \varepsilon$ the system defined by the Hamiltonian $H_0 + H_1$ has an orbit that goes from a small neighbourhood of $I = I_a$ to a small neighbourhood of $I = I_b$.

Since the unperturbed system preserves the value of I and the normally-hyperbolic cylinder is an analytic manifold foliated by the invariant circles, one can use the methods from the original Arnold paper [1] and construct a special perturbation H_1 which does not change the system on the cylinder but creates heteroclinic connections between different invariant curves on A . These connection can be used to form heteroclinic chains, and the existence of drifting orbits follows. Our statement here is, in fact, much stronger, as it guarantees the existence of the drift in I for a *typical* analytic perturbation H_1 .

A partial case is given by small periodic perturbations of autonomous Hamiltonian systems. Namely, consider a Hamiltonian of the form

$$H = H_0(p, q) + H_1(p, q, t).$$

We assume that the unperturbed Hamiltonian H_0 has, at every energy level from some interval $[I_a, I_b]$, a hyperbolic periodic orbit with a transverse homoclinic; both orbits depend continuously on the value of the energy $I = H_0$. The union of these periodic orbits over all I is a normally-hyperbolic invariant cylinder A ; the union of the homoclinics provides a homoclinic cylinder B . The twist condition reads as $\partial T / \partial I \neq 0$ where $T(I)$ is the period of the saddle periodic orbit in the energy level $H_0 = I$. For all sufficiently small H_1 which are periodic in time t the homoclinic channel $\{A, B\}$ persists and the twist condition holds.

Then our theorem implies the existence of a trajectory with the energy drifting from I_a to I_b for an open and dense (in the space of functions analytic in some complex domain Q) set of small perturbations H_1 . We note that the size of perturbation can be taken uniform for all arbitrarily large frequencies of the perturbation, therefore the result holds true in the case where Melnikov type computations are unavailable and the speed of the energy drift is bounded from above by the Nekhoroshev type estimates.

This problem is a natural extension of the Mather problem on the existence of trajectories with unbounded energy in a periodically forced geodesic flow [9, 25]. The criteria for the existence of trajectories of the energy that grows up to infinity are known [9, 25, 63, 26, 39, 40] that are based on the assumption that the initial value of energy is high enough. The above described result can be used for showing the generic existence of orbits of unbounded energy growth for all possible initial energy values.

Other type of examples to our Main Theorem is given by small perturbations of direct products of two symplectic maps. Thus, consider a 4-dimensional symplectic map which is a direct product of a twist map and a standard map: $\Phi_0 : (\varphi, I, x, y) \mapsto (\bar{\varphi}, \bar{I}, \bar{x}, \bar{y})$:

$$\begin{aligned}\bar{\varphi} &= \varphi + \omega(I), & \bar{x} &= x + \bar{y}, \\ \bar{I} &= I, & \bar{y} &= y + k \sin x,\end{aligned}\tag{2}$$

where $k > 0$ is a positive parameter and ω is an analytic function. We consider φ and x to be angular variables, so the map is a symplectic diffeomorphism of $(\mathbb{T} \times \mathbb{R})^2$. The map Φ_0 preserves the value of the I variable. Thus, the cylinder A given by $x = y = 0$ is invariant and filled with invariant curves.

The situation is more challenging when the integrable twist map is replaced by another standard map, so the new unperturbed map is given by $\Phi_0 : (\varphi, I, x, y) \mapsto (\bar{\varphi}, \bar{I}, \bar{x}, \bar{y})$:

$$\begin{aligned}\bar{\varphi} &= \varphi + \bar{I}, & \bar{x} &= x + \bar{y}, \\ \bar{I} &= I + k_1 \sin \varphi, & \bar{y} &= y + k_2 \sin x.\end{aligned}\tag{3}$$

The cylinder $A : x = y = 0$ is still invariant but it is no longer filled with invariant curves. Instead the cylinder contains a Cantor set of invariant curves provided k_1 is not too large. These tori prevent trajectories of Φ_0 from travelling in the direction of the I axis.

The theory presented in this paper allows to treat both cases equally and implies that an arbitrarily small generic analytic perturbation creates trajectories which travel between regions $I < I_a$ and $I > I_b$ for any $I_a < I_b$ (provided $\omega'(I)$ is separated from 0 for (2), and $k_2 > C(4|k_1| + k_1^2)$ for (3)). Indeed, in order to apply Theorem 2.1 to these examples, we notice, first, that the invariant cylinder A is normally hyperbolic. This cylinder has a stable and unstable separatrices $W^u(A)$ and $W^s(A)$ which coincide with the product of A and the stable (reps., unstable) separatrix of the standard map $W_{sm}^{u,s}$, so we can write (slightly abusing notation) $W^s(A) = A \times W_{sm}^s$ and $W^u(A) = A \times W_{sm}^u$. This product also describes the structure of the foliation of $W^{u,s}(A)$ into strong stable and strong unstable manifolds of points in A . For a point $v \in A$, we let $E^{uu}(v) = \{v\} \times W_{sm}^u$ and $E^{ss}(v) = \{v\} \times W_{sm}^s$. The assumption $k_2 > C(4|k_1| + k_1^2)$ for (3) ensures that these strong stable and strong unstable foliations remain C^1 -smooth after the perturbation.

It is not too difficult to prove that the standard map has infinitely many transversal homoclinic orbits for any $k > 0$. Let $\mathbf{p}_h = (x_h, y_h)$ be one of these orbits. The cylinder $B = A \times \{\mathbf{p}_h\} \subset W^u(A) \cap W^s(A)$ is homoclinic to A . Since the strong stable and strong unstable foliations of a point $v \in A$ coincide with the product of the base point and the separatrices of the standard map, we see that $(v, \mathbf{p}_h) \in E^{ss}(v) \cap E^{uu}(v)$, and the cylinder B satisfies the strong transversality assumption described in the next Section giving rise to a *simple* homoclinic intersection (defined in the Next section). Then Theorem 2.1 implies that generic perturbation of Φ_0 has orbits travelling in the direction of the cylinder A .

Similar maps were considered in Easton et al. [33] (motivated by the “stochastic pump model” of Tennyson et al. [86]). In [33] the existence of drift orbits was shown for all non-integrable Lagrangian perturbations provided k_2 is large enough (i.e. in the “anti-integrable” limit). Our methods allow to obtain the drifting orbits without the large k_2 assumption, i.e. without a detailed knowledge of the dynamics of the system. The price for that is that we do not have a constructive description of the class of perturbation which lead to the Arnold diffusion instability, although we know that these perturbations form a dense (and open) subset of the set of all analytic perturbations that keep the homoclinic channel intact.

2 Set up, assumptions, and results

Consider a real-analytic diffeomorphism $\Phi : \Sigma \rightarrow \mathbb{R}^{2d}$, $d \geq 2$, defined on an open set $\Sigma \subseteq \mathbb{R}^{2d}$. We assume that Φ preserves the standard symplectic form Ω , and that Φ is exact (the latter is always true if Σ is e.g. simply-connected). Let Φ have an invariant smooth two-dimensional cylinder A diffeomorphic to $\mathbb{S}^1 \times [0, 1]$. We denote the corresponding embedding $\mathbb{S}^1 \times [0, 1] \rightarrow \Sigma$ as ψ . The cylinder A is bounded by a pair of invariant curves $\psi(\mathbb{S}^1 \times 0)$ and $\psi(\mathbb{S}^1 \times 1)$. We will use the notation ∂A for this pair of curves, and $\text{int}(A)$ for $A \setminus \partial A$. We also denote $F_0 = \Phi|_A$.

We assume that the cylinder A is normally-hyperbolic. Namely, we assume that at each point $v \in A$ the tangent space is decomposed into direct sum of three non-zero subspaces: $\mathbb{R}^{2d} = N_v^c \oplus N_v^u \oplus N_v^s$, where N_v^c is the two-dimensional tangent to A at the point v , while $N^{s,u}$ are certain subspaces which depend continuously on v , and these subspaces fields are invariant with respect to the derivative Φ' of the map, i.e. $\Phi' N_v^s = N_{F_0(v)}^s$, $\Phi' N_v^u = N_{F_0(v)}^u$ (the condition $\Phi' N_v^c = N_{F_0(v)}^c$ is fulfilled automatically because of the invariance of A with respect to Φ). We assume that for some choice of norms in $N^{s,u,c}$ there exist $\alpha > 1$ and $\lambda \in (0, 1)$ such that at every point $v \in A$

$$\|F'_0(v)\| < \alpha, \quad \|(F'_0(v))^{-1}\| < \alpha, \quad (4)$$

$$\|\Phi'|_{N_v^s}\| < \lambda, \quad \|(\Phi'|_{N_v^u})^{-1}\| < \lambda, \quad (5)$$

and

$$\alpha^2 \lambda < 1. \quad (6)$$

Note that these assumptions are stronger compared to the standard definition of a normally-hyperbolic manifold. In particular, the large spectral gap condition (6) implies the C^1 -regularity of the strong stable and strong unstable foliations while the general case implies

Hölder continuity only (see e.g. [79]). The C^1 -regularity means, in our case, that the leaves of the foliation are smooth and, importantly, the field of tangents to the leaves is also smooth, which implies that for any two cross-sections transverse to the foliation the map between the cross-sections which is defined by the leaves of the foliation is a diffeomorphism.

We also note that, according to the above definition, A is *symmetrically* normally-hyperbolic, i.e., in (4),(5) the same pair of exponents α, λ bounds both Φ' and $(\Phi')^{-1}$. This symmetric form of the spectral gap assumption is chosen because the map Φ is symplectic. We will prove (Proposition 5.2) that the symmetric spectral gap implies that the restriction of the symplectic form on A is non-degenerate, i.e. A is a symplectic submanifold of \mathbb{R}^{2d} , therefore the map $F_0 = \Phi|_A$ inherits the (exact) symplecticity of Φ .

We are quite confident that the factors λ and α may be allowed to depend on the point v on the manifold. However, for simplicity, we conduct the proofs only for the case of v -independent λ and α .

The points in a small neighbourhood of the normally-hyperbolic cylinder A , whose forward iterations tend to A without leaving the neighbourhood, form a smooth (at least C^2 in our case) invariant manifold, the local stable manifold $W_{loc}^s \supset A$, which is tangent to $N^s \oplus N^c$ at the points of A (see e.g. [50]). The points whose backward iterations tend to A , without leaving the neighbourhood, form the local unstable smooth invariant manifold $W_{loc}^u \supset A$, which is tangent to $N^u \oplus N^c$ at the points of A . The invariant cylinder A is the intersection of W_{loc}^u and W_{loc}^s . In each of the manifolds there exists a uniquely defined C^1 -smooth invariant foliation transverse to A , the strong-stable invariant foliation E^{ss} in W_{loc}^s and the strong-unstable invariant foliation E^{uu} in W_{loc}^u , such that for every point $v \in A$ there is a unique leaf of E_v^{ss} and a unique leaf of E_v^{uu} which pass through this point and are tangent to N_v^s and, respectively, N_v^u (see [83]).

The global stable and unstable manifolds of A are defined by iterating the local invariant manifolds: $W^u(A) := \bigcup_{m \geq 0} \Phi^m W_{loc}^u$ and $W^s(A) := \bigcup_{m \geq 0} \Phi^{-m} W_{loc}^s$. As the global stable and unstable manifolds are obtained from the local ones by the iterations of Φ , the locally defined invariant foliations E^{ss} and E^{uu} are extended in a unique way on the whole of $W^s(A)$ and, respectively, $W^u(A)$.

Let us now assume that *the symmetrically normally-hyperbolic cylinder A has a homoclinic*, i.e., the intersection of $W^u(A)$ and $W^s(A)$ has a point x outside A . If $W^u(A)$ and $W^s(A)$ are transverse at x , the implicit function theorem implies that x has an open neighbourhood U_x in $W^u(A) \cap W^s(A)$, which is diffeomorphic to a two-dimensional disk.

Note that for any $x \in W^u(A) \cap W^s(A)$ there exists a unique leaf of E_x^{uu} and a unique leaf of E_x^{ss} which pass through this point. We will call the homoclinic intersection at x *strongly transverse* if

$$\mathbb{R}^{2d} = \mathcal{T}_x E_x^{ss} \oplus \mathcal{T}_x E_x^{uu} \oplus \mathcal{T}_x (W^u(A) \cap W^s(A)), \quad (7)$$

i.e. the leaf E_x^{uu} is transverse to $W^u(A)$ and the leaf E_x^{ss} is transverse to $W^s(A)$ at the point x . In other words, the foliation E^{uu} is transverse to the small disc U_x in $W^u(A)$ and the foliation E^{ss} is transverse to U_x in $W^s(A)$. Thus, the strong transversality implies that the holonomy maps $\pi^s : U_x \rightarrow A$ and $\pi^u : U_x \rightarrow A$ (projections by the leaves of the foliations E^{ss} and E^{uu} , respectively) are diffeomorphisms.

Let $\bar{A} \subset \text{int}(A)$ be a compact invariant sub-cylinder in A , i.e. it is a closed region in $\text{int}(A)$ bounded by two non-intersecting invariant essential curves γ^+ and γ^- . We say that a smooth manifold $B \subset W^u(\text{int}(A)) \cap W^s(\text{int}(A)) \setminus A$ is a *homoclinic cylinder, simple relative to the cylinder \bar{A}* , if the following assumptions hold:

[S1] The *strong transversality* condition (7) holds for all $x \in B$.

[S2] For every point $x \in \bar{A}$, the corresponding leaf of the foliation E^{uu} intersects the homoclinic cylinder B at exactly one point each, and no two points in B belong to the same leaf of the foliation E^{ss} . In other words, the projection $\pi_B^s : B \rightarrow \text{int}(A)$ by the leaves of the strong-stable foliation is injective, and the projection $(\pi_B^u)^{-1} : \bar{A} \rightarrow B$ by the leaves of the strong-unstable foliation is well-defined.

[S3] The image of \bar{A} by the map $\pi_B^s \circ (\pi_B^u)^{-1}$ contains an essential curve.

Condition [S1] implies that the projections $\pi_B^{s,u} : B \rightarrow \text{int}(A)$ are local diffeomorphisms. Condition [S2] implies that the maps $(\pi_B^s)^{-1}$ and $(\pi_B^u)^{-1}$ are well-defined (single-valued). Thus, following [28], we define the *scattering map* $F_B = \pi_B^s \circ (\pi_B^u)^{-1}$, which is a diffeomorphism $\bar{A} \rightarrow F_B(\bar{A}) \subset \text{int}(A)$. Condition [S3] implies that the image of any essential curve by the scattering map is also an essential curve, i.e. the scattering map is homotopic to identity on \bar{A} . In fact, it is easy to see that conditions [S2] and [S3] are equivalent to the requirement that the scattering map is a homotopic to identity diffeomorphism $\bar{A} \rightarrow F_B(\bar{A})$.

In particular, it follows that the set $F_B(\bar{A})$ is a cylinder bounded by two non-intersecting essential curves, $F_B(\gamma^\pm)$. We will also show (Proposition 5.5) that the scattering map F_B is an exact symplectomorphism. This implies that the cylinder $F_B(\bar{A})$ has the same area as \bar{A} and that the intersection of $F_B(\bar{A})$ and \bar{A} is non-empty.

We will call the orbit of B by the map Φ a *homoclinic sequence of cylinders*. Since the foliations E^{ss} and E^{uu} are invariant, it follows that each set $\Phi^n(B)$ in this sequence is also a simple homoclinic cylinder. Indeed, the invariance of the foliations means that $\Phi(\pi^s(x)) = \pi^s(\Phi(x))$ and $\Phi(\pi^u(x)) = \pi^u(\Phi(x))$ for every point x from $W^s(A)$ or, respectively, $W^u(A)$. Hence,

$$\pi_{\Phi(B)}^{s,u} = \Phi \circ \pi_B^{s,u} \circ \Phi^{-1}. \quad (8)$$

Thus, the scattering map satisfies

$$F_{\Phi(B)} = \Phi \circ F_B \circ \Phi^{-1}. \quad (9)$$

As we see, the scattering maps corresponding to any two different cylinders in the same homoclinic sequence are smoothly conjugate to each other, so the fulfilment of the simplicity conditions for one of the cylinders implies the fulfilment of the simplicity conditions for the other one.

Note that the fulfilment of condition [S2] depends both on the choice of the invariant cylinder \bar{A} and on the choice of the ambient invariant cylinder A : the latter must be large enough to incorporate $F_B(\bar{A})$. We will need a stronger version of this requirement. Namely,

we assume that $F_B(\bar{A}) \subseteq \hat{A}$ where $\hat{A} \supseteq \bar{A}$ is a compact invariant sub-cylinder of A . One can express this property as follows:

$$W^u(\bar{A}) \cap B \subset W^s(\hat{A}) \cap B. \quad (10)$$

We will also require that the homoclinic cylinder B is simple relative to the cylinder \hat{A} .

We further assume that one can choose symplectic coordinates (y, φ) on A such that $F_0 = \Phi|_A$ will have a twist property. Birkhoff theorem [49] implies then that the boundary curves γ^\pm of the invariant sub-cylinder \bar{A} are graphs of Lipschitz functions $y^\pm(\varphi)$.

We will study behaviour of a generic map from a small neighbourhood \mathcal{V} of the map Φ in the space of real-analytic exact symplectic maps. We define the topology in this space as follows. Take any compact neighbourhood K of the cylinder A in Σ , which includes the sequence of homoclinic cylinders, and let Q be any compact complex neighbourhood of K such that Φ is holomorphically extended onto some open neighbourhood of Q . The neighbourhood \mathcal{V} of the map Φ in the space of real-analytic exact symplectomorphisms consists of all holomorphic maps $cl(Q) \rightarrow \mathbb{C}^{2d}$ that are sufficiently close to Φ everywhere on Q and are real on K , preserve the standard symplectic form in \mathbb{R}^{2d} , and are exact; two maps belonging to this neighbourhood are close if they are uniformly close on Q .

We will assume that the invariant cylinder A persists for every map in \mathcal{V} . Namely, we assume that every map from \mathcal{V} has, in a small neighbourhood of A , a pair of non-intersecting essential invariant curves which depend continuously on the map (as continuous curves), and these curves form the boundary ∂A of the normally-hyperbolic cylinder A for our original map Φ . One shows that for every map from \mathcal{V} these curves bound a uniquely defined invariant cylinder which depends continuously on the map. We will continue to denote this cylinder as A . It is natural just to assume that A is bounded by two KAM-curves; then the required persistence of the boundary curves is guaranteed.

Note that the invariant sub-cylinder \bar{A} does not need to be preserved when the map Φ is perturbed, as the invariant curves γ^\pm that form its boundary can, in principle, be destroyed by a small perturbation (we do not assume that they are KAM-curves). However, this is not important for us (neither we require the preservation of the cylinder \hat{A}). We just define the curves $\gamma^\pm \subset A$ in an arbitrary way for all maps from \mathcal{V} so that they will depend continuously on the map, and for the map Φ they will bound the invariant cylinder \bar{A} .

We say that a map connects two open sets U^- and U^+ if the image of U^- by some forward iteration of the map intersects U^+ .

Theorem 2.1 (main theorem) *Let A be a symmetrically normally-hyperbolic compact invariant cylinder for the exact symplectic map Φ , and let $F_0 = \Phi|_A$ have a twist property. Let \bar{A} be a compact invariant sub-cylinder $\bar{A} \subset \text{int}(A)$ bounded by two non-intersecting essential curves. Let $B \subset W^u(\text{int}(A)) \cap W^s(\text{int}(A))$ be a homoclinic cylinder, simple relative to \bar{A} . Suppose $\bar{A} \subseteq \hat{A} \subseteq A$ where \hat{A} is a compact invariant sub-cylinder such that $W^u(\bar{A}) \cap B \subset W^s(\hat{A}) \cap B$ and B is simple relative to \hat{A} . Suppose A persists for every map Φ from a sufficiently small neighbourhood \mathcal{V} of Φ in the space of real-analytic exact*

symplectomorphisms $\Sigma \rightarrow \mathbb{R}^{2d}$. Let γ^- and γ^+ be simple essential curves in A , which depend continuously on the map $\tilde{\Phi}$ and which coincide with the boundary of \bar{A} when $\tilde{\Phi} = \Phi$. Then, for an open and dense subset $\tilde{\mathcal{V}}$ of \mathcal{V} , each map $\tilde{\Phi} \in \tilde{\mathcal{V}}$ connects every two open neighbourhoods U^- of γ^- and U^+ of γ^+ .

Remark 2.2 It is obvious that given any two open sets U^+ and U^- the set of maps that connect U^- and U^+ is open. We, however, show that intersection of all these sets (over all possible choices of the neighbourhoods U^- and U^+ of the given curves γ^- and γ^+) is open and dense in \mathcal{V} , i.e. given any map from \mathcal{V} there exists an open set of arbitrarily small perturbations of this map within \mathcal{V} such that each of these perturbations creates, for each pair of neighbourhoods U^- and U^+ of the curves γ^\pm , an orbit that connects U^- and U^+ .

Remark 2.3 Statements similar to Theorem 2.1 are known for non-analytic (smooth) case, see e.g. [16, 17, 74]. The main difference between the analytic and smooth case is that the class of perturbations small in the real-analytic sense is narrower than the class of perturbations that are small in the C^∞ -sense (e.g. our theorem implies the similar statement in the smooth category). Crucially, for a typical real-analytic map the normally-hyperbolic invariant cylinder A is not analytic (it has only finite smoothness), so no real-analytic perturbations can vanish on A . This makes the methods that have been used in the non-analytic case [16, 17, 74] inapplicable.

Remark 2.4 The symplectic diffeomorphism Φ can be a Poincare map for a Hamiltonian flow inside a level of constant energy. The methods of this paper can be generalised to show that if the Poincare map Φ for some Hamiltonian system satisfies the assumptions of the theorem, then a generic small perturbation of the Hamiltonian function H in the space of real-analytic Hamiltonians leads to creation of connecting orbits.

The strategy of the proof of our main theorem is as follows. We show in Proposition 3.3 that the existence of one homoclinic cylinder B which is simple relative to the two invariant cylinders \bar{A} and \hat{A} such that (10) holds implies the existence of infinitely many secondary homoclinic cylinders (all belonging to different homoclinic sequences) which are simple relative to \bar{A} . Thus, we will no longer use the existence of the invariant cylinder \hat{A} and will further consider $N \geq 8$ homoclinic cylinders B_1, \dots, B_N , all of which are simple relative to the same compact invariant cylinder \bar{A} and all belong to different homoclinic sequences, i.e. $\Phi^m(B_i) \cap B_j = \emptyset$ for all m and all $i, j = 1, \dots, N$ such that $i \neq j$. Given each of the homoclinic cylinders B_n , simple relative to the compact invariant sub-cylinder $\bar{A} \subset \text{int}(A)$, we define the corresponding scattering map $F_n : \bar{A} \rightarrow A$. By condition [S1], it is a local diffeomorphism. By condition [S2] this map is a bijection, hence it is a diffeomorphism of \bar{A} onto the set $F_n(\bar{A})$, which is a cylinder bounded by a pair of essential curves as follows from condition [S3]. Obviously, condition [S1] implies that the scattering maps are, in fact, defined in some open neighbourhood A' of \bar{A} in A .

It is a standard fact from the theory of normal hyperbolicity [50] that any strictly-invariant normally-hyperbolic compact smooth manifold with a boundary can be extended

to a locally-invariant normally-hyperbolic manifold without a boundary. In our case this means that the smooth embedding ψ that defines the invariant cylinder $A = \psi(\mathbb{S}^1 \times [0, 1])$ can be extended onto $\mathbb{S}^1 \times I$ where I is an open interval containing $[0, 1]$, and the image $\tilde{A} = \psi(\mathbb{S}^1 \times I) \supset A$ is normally-hyperbolic and locally-invariant with respect to the map Φ . Here, by the local invariance we mean that there exists a neighbourhood Z of \tilde{A} such that the iterations of each point of \tilde{A} stay in \tilde{A} until they leave Z . An important property of the locally-invariant normally-hyperbolic manifold without a boundary is that it persists at C^2 -small perturbations, i.e. all maps C^2 -close to Φ have a locally-invariant normally-hyperbolic cylinder $\tilde{A} \subset Z$ which depends on the map continuously as a C^2 -manifold¹. The continuous dependence on the map implies that the cylinder \tilde{A} remains symplectic and symmetrically normally-hyperbolic for all maps C^2 -close to Φ , e.g. for all maps from \mathcal{V} .

Note that the normal hyperbolicity implies that \tilde{A} contains all the orbits that never leave Z . In particular, any invariant curve that lies in Z must lie in \tilde{A} . Thus, the pair of simple non-intersecting essential invariant curves that, by our assumption, exist in Z for all maps from \mathcal{V} must lie in \tilde{A} . Hence, for all maps from \mathcal{V} these curves bound a compact invariant sub-cylinder $A \subset \tilde{A}$, which is uniquely defined by the choice of the invariant boundary curves and depends continuously on the map. The stable and unstable manifolds and the strong-stable and strong-unstable foliations of A also depend continuously, in the C^1 -topology, on the map. The transversality condition [S1] implies that the C^1 -smooth homoclinic cylinders B_1, \dots, B_N also persist and depend continuously on the map. Thus, the scattering maps F_1, \dots, F_N are defined on the same open subset A' of A for all maps from \mathcal{V} , and remain exact symplectomorphisms. Note that A' contains the region bounded by the curves γ^\pm .

Take any map $\tilde{\Phi} \in \mathcal{V}$. Let $(v_s)_{s=0}^m \subset A$ be an orbit of iterated function system $\{F_0, \dots, F_N\}$, i.e. for each $s = 0, \dots, m-1$ there exists $n_s = 0, \dots, N$ such that $v_{s+1} = F_{n_s}(v_s)$; if $n_s \neq 0$, then we always require $v_s \in A'$, so the corresponding map F_{n_s} is well-defined. In Section 4 we show that for any such orbit and any $\varepsilon > 0$, there is a point x_0 and a positive integer ℓ such that

$$\text{dist}(x_0, v_0) < \varepsilon, \quad \text{and} \quad \text{dist}(\tilde{\Phi}^\ell(x_0), v_m) < \varepsilon$$

(see Lemma 4.4). Note that we do not use hyperbolicity or index arguments in this lemma. We also do not use the symplecticity of the maps F_1, \dots, F_N , nor the twist property of the map F_0 . However, the fact that F_0 is an area-preserving map of a bounded invariant domain A is crucial here, as we use the Poincaré Recurrence Theorem in an essential way (we first prove a certain weak shadowing result, Lemma 4.1, that holds without any assumptions on the map F_0 , then Lemma 4.4 is deduced from it in the case of area-preserving F_0).

According to this shadowing lemma (Lemma 4.4), in order to show that two open sets are connected by the map $\tilde{\Phi}$, it is sufficient to show that the intersections of these sets with A are connected by orbits of the iterated function system $\{F_0, \dots, F_N\}$. A generalisation (Theorem 5.6) of a classical Birkhoff theorem states that if F_n , $n = 0, \dots, N$ are homotopic to identity, exact symplectomorphisms, and F_0 is a twist map, then for any two essential

¹Throughout this paper we assume the large spectral gap assumption (6) in the notion of normal hyperbolicity. This guarantees the C^2 -smoothness of the manifold, and the C^1 -smoothness of the corresponding strong-stable and strong-unstable invariant foliations for every map C^2 -close to Φ .

curves $\gamma^\pm \subset A'$ there is a trajectory of the iterated function system with $v_0 \in \gamma^-$ and $v_m \in \gamma^+$ unless the functions F_n have a common invariant essential curve.

Thus, if the maps F_0, \dots, F_N have no common invariant essential curves between γ^- and γ^+ , every pair of neighbourhoods, U^- of γ^- and U^+ of γ^+ , is connected by the map $\tilde{\Phi}$. Theorem 5.6 also implies that this property of the absence of a common invariant essential curve is open.

Theorem 6.1 establishes that this property is also dense in \mathcal{V} (provided $N \geq 8$). Thus, for every map $\tilde{\Phi}$ from an open and dense subset of \mathcal{V} , the corresponding scattering maps F_1, \dots, F_N ($N \geq 8$) and F_0 do not have any common essential invariant curve. As we just explained, this implies that every two neighbourhoods U^\pm of γ^\pm are connected by each such map $\tilde{\Phi}$, and Theorem 2.1 follows.

Theorem 6.1 is the crucial step in the proof of main theorem 2.1. An analogue of Theorem 6.1 for generic non-analytic maps can be derived from [16, 17, 74]. However, the methods of those papers cannot be used in the analytic case (as the real-analytic perturbations cannot, in general, vanish on the finitely smooth normally-hyperbolic cylinder). Therefore, we develop a completely different perturbation technique in order to prove Theorem 6.1 for the analytic case.

3 Estimates in a neighbourhood of a symmetrically normally-hyperbolic invariant cylinder

3.1 Fenichel coordinates, cross form of the map, and estimates for the local dynamics

We start with the analysis of the behaviour in a small neighbourhood of a normally-hyperbolic cylinder. We do not need analyticity or symplecticity of the map in this and the next Sections.

Let A be a compact, symmetrically normally-hyperbolic, smooth, invariant cylinder of a C^r -smooth map Φ ($r \geq 2$). As we mentioned, A can be extended to a larger, smooth normally-hyperbolic locally-invariant cylinder \tilde{A} . Let us introduce coordinates in a small neighbourhood of A such that this larger invariant cylinder is straightened; moreover, the local stable and unstable manifolds $W_{loc}^{s,u}$ are straightened as well, along with the strong-stable and strong-unstable foliations E^{ss} and E^{uu} on them. Note that the foliations are at least C^1 . Thus, the straightening of the foliations means that one can introduce C^1 -coordinates (u, v, z) in a neighbourhood of A such that the manifold W_{loc}^s will have equation $z = 0$, the manifold W_{loc}^u will be given by $u = 0$, and the leaves of the foliations E^{ss} and E^{uu} will all have the form $\{z = 0, v = \text{const}\}$ and, respectively, $\{u = 0, v = \text{const}\}$ (cf. [52]). The cylinder A thus lies in $(u = 0, z = 0)$. Here $v = (\varphi, y)$ with $\varphi \in \mathbb{S}^1$ being the angular variable, and y taking values from some (maybe φ -dependent) interval I of the real line.

Note that the manifolds $W^u(A)$ and $W^s(A)$ can be non-orientable, so when we say we introduce global coordinates (u, v, z) in a neighbourhood of A , we mean that $\varphi = 0$ and

$\varphi = 2\pi$ are glued by means of some linear involution in the space of coordinates (v, z) . This does not affect any estimates below.

Since the manifolds and foliations under consideration are invariant with respect to the map Φ , it follows that in our coordinates the map near A has the form

$$\bar{u} = h_1(u, z, v), \quad \bar{z} = h_2(u, z, v), \quad \bar{v} = F_0(v) + h_3(u, z, v), \quad (11)$$

where $h_{1,2,3}$ and F_0 are C^1 -functions such that

$$\begin{aligned} h_1(0, z, v) &\equiv 0, & h_2(u, 0, v) &\equiv 0, \\ h_3(0, z, v) &\equiv 0, & h_3(u, 0, v) &\equiv 0. \end{aligned} \quad (12)$$

It is seen from (11),(12) that the manifolds $u = 0$ and $z = 0$, and the foliations $(v = \text{const}, u = 0)$ and $(v = \text{const}, z = 0)$ are indeed invariant with respect to the map. Note that by construction

$$\left. \frac{\partial h_1}{\partial u} \right|_{z=0} = \Phi'|_{N^s}, \quad \left. \frac{\partial h_2}{\partial z} \right|_{u=0} = \Phi'|_{N^u},$$

therefore, we obtain from (5) that (for an appropriate choice of norms)

$$\left\| \frac{\partial h_1}{\partial u} \right\| < \lambda, \quad \left\| \left(\frac{\partial h_2}{\partial z} \right)^{-1} \right\| < \lambda.$$

Using the implicit function theorem for small u and z , one may resolve the \bar{z} -equation in (11) with respect to z . Therefore, the map $\Phi : (u, v, z) \mapsto (\bar{u}, \bar{v}, \bar{z})$ can be written, in some neighbourhood of the closed invariant cylinder A in the following cross form:

$$\bar{u} = p(u, v, \bar{z}), \quad z = q(u, v, \bar{z}), \quad (13)$$

$$\bar{v} = F_0(v) + f(u, v, \bar{z}), \quad (14)$$

such that

$$p(0, v, \bar{z}) \equiv 0, \quad q(u, v, 0) \equiv 0, \quad (15)$$

$$f(0, v, \bar{z}) \equiv 0, \quad f(u, v, 0) \equiv 0, \quad (16)$$

$$\|F'_0(v)\| < \alpha, \quad \|(F'_0(v))^{-1}\| < \alpha, \quad (17)$$

$$\left\| \frac{\partial p}{\partial u} \right\| < \lambda, \quad \left\| \frac{\partial q}{\partial \bar{z}} \right\| < \lambda, \quad (18)$$

where

$$\alpha\lambda < 1, \quad 0 < \lambda < 1 < \alpha, \quad (19)$$

see (4)-(6). Let Z_δ denote a δ -neighbourhood of A .

Lemma 3.1 *There is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and any $k \geq 0$ the following statements hold.*

1. Any trajectory of length k such that $(u_i, v_i, z_i) := \Phi^i(u_0, v_0, z_0) \in Z_\delta$ for $i = 0, \dots, k$ satisfies the following estimates for $i = 0, \dots, k$:

$$\|u_i\| \leq \delta \lambda^i, \quad \|z_i\| \leq \delta \lambda^{k-i}, \quad (20)$$

$$\|v_i - F_0^i v_0\| \leq \delta (\alpha \lambda)^{k/2}, \quad \|v_i - F_0^{i-k} v_k\| \leq \delta (\alpha \lambda)^{k/2}. \quad (21)$$

2. The orbit (u_i, v_i, z_i) is determined in a unique way for any given u_0, v_0, z_k such that $\|u_0, z_k\| \leq \delta$ and $v_0 \in A$, as well as for any given u_0, v_k, z_k such that $\|u_0, z_k\| \leq \delta$ and $v_k \in A$.

3. Moreover, as $k \rightarrow +\infty$,

$$\left\| \frac{\partial z_0}{\partial(u_0, v_0)} \right\| + \left\| \frac{\partial(u_k, v_k)}{\partial z_k} \right\| \rightarrow 0, \quad \left\| \frac{\partial u_k}{\partial(v_k, z_k)} \right\| + \left\| \frac{\partial(v_0, z_0)}{\partial u_0} \right\| \rightarrow 0, \quad (22)$$

uniformly for all $\|u_0, z_k\| \leq \delta$ and all $v_k \in A$ or $v_0 \in A$.

4. We also have for all k large enough

$$\left\| \frac{\partial z_0}{\partial z_k} \right\| \leq \lambda^k, \quad \left\| \frac{\partial(u_k, v_k)}{\partial(u_0, v_0)} \right\| \leq \alpha^k \quad (23)$$

(at any given (u_0, v_0) in the first inequality, and at any given z_k in the second one), and

$$\left\| \frac{\partial u_k}{\partial u_0} \right\| \leq \lambda^k, \quad \left\| \frac{\partial(v_0, z_0)}{\partial(v_k, z_k)} \right\| \leq \alpha^k, \quad (24)$$

(at any given (v_k, z_k) in the first inequality, and at any given u_0 in the second one).

Proof. Using (13), we get

$$u_{i+1} = p(u_i, v_i, z_{i+1}), \quad z_i = q(u_i, v_i, z_{i+1}), \quad v_{i+1} = F_0(v_i) + f(u_i, v_i, z_{i+1}), \quad (25)$$

for all $i = 0, \dots, k-1$. Equations (15) and (18) imply that

$$\|u_{i+1}\| = \|p(u_i, v_i, z_{i+1})\| \leq \lambda \|u_i\|, \quad \|z_i\| = \|q(u_i, v_i, z_{i+1})\| \leq \lambda \|z_{i+1}\|. \quad (26)$$

Since $\|u_0\|, \|z_k\| \leq \delta$, it follows that the orbit $\{(u_i, z_i, v_i)\}_{i=0}^k$ satisfies (20).

For the future convenience let us define

$$C_0(\delta) = \max \left\{ \sup_{Z_\delta} \|p'_v\|, \sup_{Z_\delta} \|p'_z\|, \sup_{Z_\delta} \|q'_u\|, \sup_{Z_\delta} \|q'_z\|, \sup_{Z_\delta} \|f'_u\|, \sup_{Z_\delta} \|f'_v\|, \sup_{Z_\delta} \|f'_z\| \right\}, \quad (27)$$

and note that $C_0(\delta)$ can be made as small as we need by decreasing δ because (15) and (16) imply that $p'_v = 0, p'_z = 0, q'_u = 0, q'_v = 0, f'_u = 0, f'_v = 0, f'_z = 0$ at $(u = 0, z = 0)$, for all $v \in A$.

In order to establish inequalities (21), let $V_i := v_i - F_0^i v_0$. In particular $V_0 = 0$. Equation (25) implies

$$\|V_{i+1}\| \leq \sup_{v \in A} \|F_0'(v)\| \cdot \|V_i\| + \|f(u_i, v_i, z_{i+1})\|. \quad (28)$$

Then equation (16) implies

$$\begin{aligned} \|f(u_i, v_i, z_{i+1})\| &\leq \sup_{(u,v,z) \in Z_\delta} \|f'_u\| \cdot \|u_i\| \leq C_0(\delta) \|u_i\|, \\ \|f(u_i, v_i, z_{i+1})\| &\leq \sup_{(u,v,z) \in Z_\delta} \|f'_z\| \cdot \|z_{i+1}\| \leq C_0(\delta) \|z_{i+1}\|. \end{aligned}$$

Thus, by (20),

$$\|f(u_i, v_i, z_{i+1})\| \leq \delta C_0(\delta) \min\{\lambda^i, \lambda^{k-i-1}\}. \quad (29)$$

Now, by (17) and (29), we may rewrite (28) as

$$\|V_{i+1}\| \leq \alpha \|V_i\| + \delta C_0(\delta) \min\{\lambda^i, \lambda^{k-i-1}\}.$$

Since $V_0 = 0$, we find (see (19)) that for all $1 \leq j \leq k$

$$\begin{aligned} \|V_j\| &\leq \delta C_0(\delta) \sum_{0 \leq i \leq j-1} \alpha^{j-i-1} \min\{\lambda^i, \lambda^{k-i-1}\} \leq \delta C_0(\delta) \sum_{0 \leq i \leq k-1} \alpha^{k-i-1} \min\{\lambda^i, \lambda^{k-i-1}\} = \\ &= \delta C_0(\delta) \left\{ \sum_{0 \leq i \leq (k-1)/2} (\alpha\lambda)^{k-i-1} + \alpha^{k-1} \sum_{(k-1)/2 < i \leq k-1} (\lambda/\alpha)^i \right\} \leq \delta(\alpha\lambda)^{k/2}, \end{aligned}$$

when δ_0 is chosen small enough to ensure $\frac{C_0(\delta)}{\sqrt{\alpha\lambda}} \left[\frac{1}{1-\alpha\lambda} + \frac{\lambda}{\alpha-\lambda} \right] \leq 1$. The first of inequalities (21) is proved. The second inequality follows immediately by the symmetry of the problem (if we replace the map Φ by its inverse, then F_0 changes to F_0^{-1} , i to $(k-i)$, (u_0, z_k) to (z_k, u_0) and v_0 to v_k).

Given u_0, v_0, z_k , the orbit $\{(u_i, z_i, v_i)\}_{i=0}^k$ is a fixed point of the operator

$$Q : \{(u_i, v_i, z_i)\}_{i=0}^k \mapsto \{(\hat{u}_i, \hat{v}_i, \hat{z}_i)\}_{i=0}^k,$$

which acts on a sequence $\{(u_i, v_i, z_i)\}_{i=0}^k$ by

$$\begin{cases} \hat{u}_{i+1} = p(u_i, v_i, z_{i+1}), & \hat{z}_i = q(u_i, v_i, z_{i+1}), \\ \hat{v}_{i+1} = F_0(v_i) + f(u_i, v_i, z_{i+1}) & \text{for } i = 0, \dots, k-1, \\ \hat{u}_0 = u_0, & \hat{v}_0 = v_0, & \hat{z}_k = z_k. \end{cases} \quad (30)$$

Recall that $v = (y, \varphi)$, where $\varphi \in \mathbb{S}^1$, and y runs an interval I such that for all sufficiently small δ the points in the δ -neighbourhood Z_δ of the cylinder A have the y -coordinates strictly inside I . It will, however, be convenient to extend the functions p, q, F_0, f in (13), (14) to all $y \in \mathbb{R}^1$ in such a way that they will remain smooth, will have uniformly continuous

derivatives, identities (15),(16) will hold, and estimates (17),(18) will stay true with a margin of safety. We assume that this extension is done without changing the functions p, q, F_0, f inside the interval I of y -values. Thus, if for a fixed point of the operator Q given by (30) with the functions p, q, F_0, f , which are defined now for all y , every (u_i, v_i, z_i) , $i = 0, \dots, k$, belongs to Z_δ , then this fixed point is an orbit of the original map Φ .

It is also convenient to consider the lift of the original map so that φ runs the whole real axis and the functions p, q and $F_0 + f - v$ are periodic in φ . So, in the analysis of the operator Q given by (30), we assume $v \in \mathbb{R}^2$.

Denote by $X = X_{k, u_0, v_0, z_k}$ the set of all sequences $\{(u_i, v_i, z_i)\}_{i=0}^k$ with the given value of (u_0, v_0, z_k) , which also satisfy $\|u_i, z_i\| \leq \delta$ for all $i = 0, \dots, k$. By (26), if $\|u_i, z_i\| \leq \delta$ for all $i = 0, \dots, k$, then $\|\hat{u}_i, \hat{z}_i\| \leq \delta$ for all $i = 0, \dots, k$ as well, thus $QX \subseteq X$. Let us show that the operator Q is contracting on X in the norm

$$\|\{(u_i, v_i, z_i)\}_{i=0}^k\|_\alpha = \max_{i=0, \dots, k} \alpha^{-i} \|u_i, v_i, z_i\|.$$

Indeed, in this norm

$$\begin{aligned} \|Q'\|_\alpha &\leq \max \left\{ \alpha^{-1} \left\| \frac{\partial p}{\partial u} \right\| + \alpha^{-1} \left\| \frac{\partial p}{\partial v} \right\| + \left\| \frac{\partial p}{\partial \bar{z}} \right\|, \quad \left\| \frac{\partial q}{\partial u} \right\| + \left\| \frac{\partial q}{\partial v} \right\| + \alpha \left\| \frac{\partial q}{\partial \bar{z}} \right\|, \right. \\ &\quad \left. \alpha^{-1} \|F'_0\| + \alpha^{-1} \left\| \frac{\partial f}{\partial u} \right\| + \alpha^{-1} \left\| \frac{\partial f}{\partial v} \right\| + \left\| \frac{\partial f}{\partial \bar{z}} \right\| \right\} \leq \\ &\leq \max \left\{ \alpha^{-1} \lambda + \alpha^{-1} C_0(\delta) + C_0(\delta), \quad 2C_0(\delta) + \alpha \lambda, \quad \alpha^{-1} \|F'_0\| + \alpha^{-1} C_0(\delta) + 2C_0(\delta) \right\}, \end{aligned}$$

where, for the derivatives in the right-hand side, we use the supremum norm taken over all (u, v, \bar{z}) such that $\|u, \bar{z}\| \leq \delta$, and $C_0(\delta)$ is defined by (27). By (15)-(19), if δ is sufficiently small, then $\|Q'\|_\alpha < 1$ uniformly for every element from X , independently of the value of $k \geq 0$. Since the set X is convex, it follows that the operator Q is indeed contracting.

Thus, by contraction mapping principle, given any (u_0, v_0, z_k) such that $\|u_0, z_k\| \leq \delta$ there exists indeed a unique length- k orbit with the given values of u_0, v_0 and z_k . We already proved that this orbit must satisfy (20) and (21). Since $v_0 \in A$ implies $F_0^i v_0 \in A$ for all $i = 0, \dots, k$ by the invariance of A with respect to F_0 , estimates (20) and (21) imply that the orbit lies in Z_δ as required.

By the symmetry of the problem, given any (u_0, v_k, z_k) such that $\|u_0, z_k\| \leq \delta$ and $v_k \in A$, there, as well, exists a unique length- k orbit with the given values of u_0, v_k and z_k , and this orbit lies in Z_δ .

As a fixed point of a smooth contracting operator, the obtained orbit must depend smoothly on all data on which the operator depends smoothly. So, (u_i, v_i, z_i) depend smoothly on (u_0, v_0, z_k) (and, by the symmetry of the problem, on (u_0, v_k, z_k) as well). To complete the proof of the lemma, it remains to prove estimates (22), (23) and (24).

We will prove only the first limit in (22), as the second one follows then due to the symmetry of the problem with respect to change of Φ to Φ^{-1} ; it is also enough to prove only (23), as (24) follows then by the symmetry. Denote $\beta_i = \|\partial(u_i, v_i)/\partial(u_0, v_0)\|$, $\gamma_i =$

$\|\partial z_i/\partial(u_0, v_0)\|$, where the derivatives are taken at z_k fixed. By differentiating (25), we obtain

$$\beta_{i+1} \leq \left\| \frac{\partial(p, F_0 + f)}{\partial(u, v)} \right\| \beta_i + \left\| \frac{\partial(p, f)}{\partial \bar{z}} \right\| \gamma_{i+1}, \quad \gamma_i \leq \left\| \frac{\partial q}{\partial \bar{z}} \right\| \gamma_{i+1} + \left\| \frac{\partial q}{\partial(u, v)} \right\| \beta_i,$$

where the derivatives are taken at $(u, v, \bar{z}) = (u_i, v_i, z_{i+1})$. Since u_i and z_i satisfy (20), we obtain from (15)-(18) that for sufficiently small δ (independent of i and k)

$$\beta_{i+1} \leq (\alpha - \rho)\beta_i + \mu_i \gamma_{i+1}, \quad \gamma_i \leq (\lambda - \rho)\gamma_{i+1} + \mu_{k-i-1}\beta_i, \quad (31)$$

where ρ is a small positive constant, and

$$\mu_j = \sup_{\|u\| \leq \delta \lambda^j, (u, v, z) \in Z_\delta} \left\| \frac{\partial(p, f)}{\partial \bar{z}} \right\| + \sup_{\|z\| \leq \delta \lambda^j, (u, v, z) \in Z_\delta} \left\| \frac{\partial q}{\partial(u, v)} \right\|; \quad (32)$$

it follows from (15),(16) that

$$\mu_j \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (33)$$

Recall also that, by definition,

$$\beta_0 = 1, \quad \gamma_k = 0. \quad (34)$$

Define the sequence M_j by the rule

$$M_{j+1} = \alpha \lambda M_j + \mu_j, \quad (35)$$

for an arbitrarily chosen M_0 . As $\alpha \lambda < 1$, it follows from (33) that

$$M_j \rightarrow 0 \quad \text{as } j \rightarrow +\infty. \quad (36)$$

By (31)

$$\gamma_i - M_{k-i}\beta_i \leq \frac{\lambda - \rho}{1 - \mu_i M_{k-i-1}} (\gamma_{i+1} - M_{k-i-1}\beta_{i+1}) + \left[\mu_{k-i-1} - M_{k-i} + \alpha \frac{\lambda - \rho}{1 - \mu_i M_{k-i-1}} M_{k-i-1} \right] \beta_i.$$

As both the sequences μ_j and M_j tend to zero, it follows that

$$\lim_{k \rightarrow +\infty} \max_{i=0, \dots, k-1} \mu_i M_{k-i-1} = 0, \quad (37)$$

so if k is large enough, then $\mu_i M_{k-i-1} < \rho/\lambda < 1$ for all $i = 0, \dots, k-1$. Thus,

$$\gamma_i - M_{k-i}\beta_i \leq \lambda(\gamma_{i+1} - M_{k-i-1}\beta_{i+1}) + [\mu_{k-i-1} - M_{k-i} + \alpha \lambda M_{k-i-1}] \beta_i,$$

which, by (35), implies

$$\gamma_i - M_{k-i}\beta_i \leq \lambda(\gamma_{i+1} - M_{k-i-1}\beta_{i+1}),$$

hence, for all k large enough, for every $i = 0, \dots, k-1$

$$\gamma_i - M_{k-i}\beta_i \leq \lambda^{k-i}(\gamma_k - M_0\beta_k), \quad (38)$$

in particular

$$\gamma_0 - M_k \beta_0 \leq \lambda^k (\gamma_k - M_0 \beta_k). \quad (39)$$

Now, by (34), we have $\gamma_0 \leq M_k$, so (36) implies $\partial z_0 / \partial (u_0, v_0) \rightarrow 0$ as $k \rightarrow +\infty$, which agrees with (22). Note also that by (38) we have $\gamma_{i+1} \leq M_{k-i-1} \beta_{i+1}$. By (31), (37), this gives us that for all k large enough, for every $i = 0, \dots, k-1$

$$\beta_{i+1} \leq \alpha \beta_i,$$

which (see (34)) implies the second inequality in (23).

It remains to estimate $\partial(u_k, v_k, z_0) / \partial z_k$ as $k \rightarrow +\infty$. To this aim, we denote now $\beta_i = \|\partial(u_i, v_i) / \partial z_k\|$, $\gamma_i = \|\partial z_i / \partial z_k\|$, where the derivatives are taken at (u_0, v_0) fixed. Then by differentiating (13), (14), we will obtain exactly the same inequalities (32) as before, hence the estimate (39) holds at all sufficiently large k for the newly defined γ_i, β_i . However, instead of (34) we have now

$$\beta_0 = 0, \quad \gamma_k = 1.$$

Thus, we find from (39) that

$$\gamma_0 \leq \lambda^k, \quad \beta_k \leq 1/M_0$$

for all k sufficiently large. This immediately gives us the first inequality in (23), and since M_0 can be taken arbitrary, we also obtain that $\partial(u_k, v_k) / \partial z_k \rightarrow 0$ as $k \rightarrow +\infty$, which finishes the proof of (22). \square

3.2 “Lambda-lemma”

The following analogue of the “lambda-lemma” [76, 20] is extracted from Lemma 3.1.

Proposition 3.2 *Given any surface L of the form $u = w(v, z)$, where w is a smooth function defined for all $v \in A$ and all small z , the images of L by the map Φ converge to $W_{loc}^u(A)$ in the C^1 -topology. If a surface L has the form $z = w(v, u)$, where w is a smooth function defined for all $v \in A$ and all small u , then the images $\Phi^{-m}L$ converge (in C^1) to $W_{loc}^s(A)$ as $m \rightarrow +\infty$*

Proof. By the symmetry of the problem, it is enough to consider only the case where L is a surface of the form $u = w(v, z)$. By Lemma 3.1, given any (u_0, v_k, z_k) the corresponding orbit (u_i, v_i, z_i) is defined uniquely. Denote as η_k the operator that sends (u_0, v_k, z_k) to (v_0, z_0) , and as ξ_k the operator that sends (u_0, v_k, z_k) to u_k . The point (u, v, z) belongs to $\Phi^k L$ if and only if $u_0 = w(v_0, z_0)$, i.e. the equation of $\Phi^k L$ is

$$u_k = \xi_k(u_0, v_k, z_k) \quad (40)$$

where u_0 is defined from

$$u_0 = w(\eta_k(u_0, v_k, z_k)). \quad (41)$$

By (20) and (22),

$$\|\eta_k\| + \|\partial \eta_k / \partial u_0\| \rightarrow 0 \text{ as } k \rightarrow +\infty,$$

therefore at each k large enough equation (41) defines u_0 uniquely as a smooth function of (v_k, z_k) . It follows from (24) that

$$\|\partial u_0 / \partial (v_k, z_k)\| = O(\alpha^k).$$

Thus, equation (40) defines u_k as a smooth function $w_k(v_k, z_k)$, for all $\|z_k\| \leq \delta$ and $v_k \in A$. By (20), $\|u_k\| \rightarrow 0$ as $k \rightarrow +\infty$. Moreover, since by (24) and (22) we have $\|\partial \xi_k / \partial u_0\| = O(\lambda^k)$ and $\|\partial \xi_k / \partial (v_k, z_k)\| \rightarrow 0$ as $k \rightarrow +\infty$, it follows that

$$\left\| \frac{dw_k}{d(v_k, z_k)} \right\| \leq \left\| \frac{\partial \xi_k}{\partial u_0} \right\| \cdot \left\| \frac{\partial u_0}{\partial (v_k, z_k)} \right\| + \left\| \frac{\partial \xi_k}{\partial (v_k, z_k)} \right\| = O((\alpha\lambda)^k) + \left\| \frac{\partial \xi_k}{\partial (v_k, z_k)} \right\| \rightarrow 0$$

as $k \rightarrow +\infty$ (recall that $\alpha\lambda < 1$). We see that for all k large enough the surface $\Phi^k L$ is given by the equation $u = w_k(v, z)$ where w_k tends to zero along with the first derivative as $k \rightarrow +\infty$. Since equation of W_{loc}^u is $u = 0$, this proves the proposition. \square

3.3 Secondary homoclinic cylinders

Proposition 3.2 allows to establish a sufficient condition for the existence of infinitely many homoclinic cylinders. Let $\bar{A} \subset A$ and $\hat{A} \subset A$ be a pair of compact invariant cylinders, $\bar{A} \subset \hat{A}$. Let the intersection of $W^u(A)$ and $W^s(A)$ be non-empty and contain a homoclinic cylinder B which is simple relative to \hat{A} . Since the manifolds $W^u(A)$ and $W^s(A)$ are invariant with respect to Φ , all iterations $\Phi^m B$ also belong to the intersection $W^u(A) \cap W^s(A)$, and this homoclinic sequence of cylinders converges to A as $m \rightarrow \pm\infty$. As we explained before (see (9)), if B is a simple homoclinic cylinder relative to \bar{A} then the cylinders $\Phi^m B$, for any m , also has this property.

Proposition 3.3 *Let condition (10) hold for the pair of cylinders \bar{A} and \hat{A} . Then there are infinitely many homoclinic cylinders B_i , each corresponds to a simple (relative to \bar{A}) intersection of $W^u(A)$ with $W^s(A)$, and none of the cylinders belongs to the homoclinic sequence corresponding to another cylinder: $B_i \cap \Phi^m B_j = \emptyset$ for every m and every $i \neq j$.*

Proof. Take the homoclinic cylinder B and consider its iterations $B^- = \Phi^{-m_-}(B)$ and $B^+ = \Phi^{m_+}(B)$ ($m_{\pm} > 0$) which lie in a small neighbourhood of A . Thus, $B^- \subset W_{loc}^u$ and $B^+ \subset W_{loc}^s$. By the relative to \hat{A} simplicity of B , the cylinder B^- intersects transversely the leaves of E^{uu} in $W_{loc}^u(\hat{A})$ at a single point each. Since W_{loc}^u has the equation $u = 0$ and the leaves of the foliation E^{uu} in W_{loc}^s are given by $\{u = 0, v = \text{const}\}$, it follows that there is a piece W of the manifold $W^s(A)$ which contains the homoclinic cylinder B^- and has the form $z = w(v, u)$ where w is a smooth function defined for all v from some neighbourhood of \hat{A} and all small u . By Proposition 3.2 (where the invariant cylinder A is replaced by the invariant cylinder \hat{A}), the images $W_i = \Phi^{-i}W$ by the backward iterations of Φ accumulate on $W_{loc}^s(\hat{A})$ in C^1 . By (10), $W_{loc}^s(\hat{A}) \cap B^+ \supset W^u(\bar{A}) \cap B^+$. Therefore, each of W_i with i sufficiently large intersects $W^u(\bar{A})$ near B^+ transversely. Since W_i are, by construction, pieces of $W^u(A)$, this gives us the sought infinite set of homoclinic cylinders B_i (converging

to the cylinder B^+). Since W_i are C^1 -close to $W_{loc}^s(\hat{A})$ near B^+ , it follows from the relative to \hat{A} simplicity of B^+ that W_i intersect transversely each leaf of the foliation E^{uu} in $W^u(\bar{A})$, the uniqueness of the intersections is also inherited.

Thus, the scattering maps $F_i : \bar{A} \rightarrow \text{int}(A)$ are defined for each of the cylinders B_i . In order to check the simplicity of the homoclinic intersection at B_i , we need to show that the projections $\pi_{B_i}^s : B_i \rightarrow \text{int}(A)$ by the leaves of the strong-stable foliation are injective for all i (condition [S2]), and that the scattering maps are homotopic to identity (condition [S3]). To check the injectivity, notice that

$$\pi_{\Phi^i(B_i)}^s = \Phi^i \circ \pi_{B_i}^s \circ \Phi^{-i} \quad (42)$$

by (8). So, it is enough to show the injectivity of $\pi_{\Phi^i(B_i)}^s$. To do this, note that the cylinders $\Phi^i(B_i)$ are close to B^- at large i , so the maps $\pi_{\Phi^i(B_i)}^s$ are close to $\pi_{B^-}^s$, and the latter map is injective by the simplicity of the homoclinic intersection at B .

It remains to show that the scattering maps F_i are homotopic to identity. As we just mentioned, the maps $\hat{\pi}_i^s = \pi_{\Phi^i(B_i)}^s \circ (\pi_{B^-}^s)^{-1}$ are close to identity at large i . The same is true for the maps $\hat{\pi}_i^u = \pi_{B^+}^u \circ (\pi_{B_i}^u)^{-1}$. Using (42), we find

$$F_i = \pi_{B_i}^s \circ (\pi_{B_i}^u)^{-1} = \Phi^i \circ \hat{\pi}_i^s \circ F_{B^-} \circ \pi_{B^-}^u \circ \Phi^{-i} \circ (\pi_{B^+}^s)^{-1} \circ F_{B^+} \circ \hat{\pi}_i^u, \quad (43)$$

where $F_{B^+} = \pi_{B^+}^s \circ (\pi_{B^+}^u)^{-1}$ and $F_{B^-} = \pi_{B^-}^s \circ (\pi_{B^-}^u)^{-1}$ are the scattering maps corresponding to the cylinders B^+ and B^- . By the simplicity of the homoclinic intersection at B , these maps are homotopic to identity diffeomorphisms. The map Φ^i in formula (43) acts in a small neighbourhood Z of A and is homotopic to identity in Z . The maps $\pi_{B^+}^s$ and $\pi_{B^-}^u$ are projections along the foliations in the local stable and unstable manifolds, so they are homotopic to identity in Z . Thus, all the maps in the right-hand side of formula (43) are homotopic to identity, which implies that the scattering maps F_i are homotopic to identity for all i large enough. This proves the claim. \square

This proposition shows that the assumptions of Theorem 2.1 imply the existence of an infinite series of homoclinic cylinders which are simple relative to \bar{A} . For our purposes, the existence of $N \geq 8$ such cylinders is enough, so it will be our standing assumption for the rest of the paper. We do not need the auxilliary invariant cylinder \hat{A} anymore.

4 Shadowing in the homoclinic channel

4.1 Homoclinic channel

Let B_1, \dots, B_N be homoclinic cylinders, each corresponds to a simple homoclinic intersection relative to the compact invariant subcylinder \bar{A} , and none of the cylinders B_n belongs to the homoclinic sequence corresponding to another cylinder. Let us repeat the definition of the scattering maps F_n . Since the homoclinic intersections are simple, it follows that two diffeomorphisms, π_n^u and π_n^s , from B_n into A are defined for every n by the leaves of the foliations E^{uu} and E^{ss} , respectively. Namely, $v = \pi_n^u(x)$ if the points $x \in B_n$ and $v \in A$

belong to the same leaf of the foliation E^{uu} , and $v = \pi_n^s(x)$ if $x \in B_n$ and $v \in A$ belong to the same leaf of the foliation E^{ss} (the smoothness of the maps π_n^s and π_n^u and their inverse maps follows from the transversality of the intersections of the leaves with B_n). By assumption, $\bar{A} \subset \pi_n^s(B_n)$. Thus, for each homoclinic cylinder B_n we have a diffeomorphism $F_n = \pi_n^s \circ (\pi_n^u)^{-1}$ which acts from \bar{A} into A . In fact, as the strong transversality condition [S1] is open, there is a neighbourhood A' of \bar{A} such that all the scattering maps F_1, \dots, F_N are diffeomorphisms of A' into A .

Take sufficiently large positive m_+ and m_- such that all the cylinders $B_n^+ = \Phi^{m_+}(B_n)$ and $B_n^- = \Phi^{-m_-}(B_n)$ ($n = 1, \dots, N$) lie in the δ -neighbourhood of A , where δ is small enough. As $B_n^+ \in W_{loc}^s$ and $B_n^- \in W_{loc}^u$, it follows that $z = 0$ on B_n^+ , and $u = 0$ on B_n^- . Since the homoclinics are simple, the cylinder B_n^+ intersects the leaves $\{v = \text{const}\}$ of the foliation E^{ss} in W_{loc}^s transversely, no more than at one point each, hence B_n^+ is given by $B_n^+ : \{u = u_n^+(v), z = 0\}$, where u^+ is a smooth function whose domain of definition contains \bar{A} . Analogously, $B_n^- : \{z = z_n^-(v), u = 0\}$ for a smooth function z^- . Thus, the points on B_n^+ and B_n^- are uniquely determined by their v -coordinates. This gives us a trivial projection of B_n^+ and B_n^- to A , so we may formally consider the maps F_n , $n = 0, \dots, N$, as acting on B_n^- or B_n^+ in the same way these maps act on A .

Since the foliations E^{ss} and E^{uu} are invariant with respect to the map Φ , it follows that $\Phi(E_v^{uu}) = E_{F_0(v)}^{uu}$ and $\Phi(E_v^{ss}) = E_{F_0(v)}^{ss}$ and, consequently, the points $F_0^{-m_-} \circ \pi_n^u(x)$ and $\Phi^{-m_-}(x)$ have the same v -coordinate given any $x \in B_n$, and the same is true for the points $F_0^{m_+} \circ \pi_n^s(x)$ and $\Phi^{m_+}(x)$. Thus, in the v -coordinates, we have

$$\Phi^{m_++m_-}|_{B_n^-} = F_0^{m_+} \circ F_n \circ F_0^{m_-} \quad (44)$$

Denote as $T_n : (u, v, z) \mapsto (\bar{u}, \bar{v}, \bar{z})$ the map $\Phi^{m_++m_-}$ from a small neighbourhood of B_n^- to a small neighbourhood of B_n^+ . The transversality condition implies that the image by the map T_n of any leaf ($u = 0, v = \text{const}$) (of the foliation E^{uu} in W_{loc}^u) is transverse to $W_{loc}^s : \{\bar{z} = 0\}$, which means that the derivative $\partial \bar{z} / \partial z$ is invertible. Therefore, given any small (u, \bar{z}) and $v \in \bar{A}$ we have a uniquely defined (\bar{u}, z, \bar{v}) such that $(\bar{u}, \bar{z}, \bar{v}) = T_n(u, z, v)$. So, we may write the map T_n in the following form:

$$\bar{u} = p_n(u, v, \bar{z}), \quad \bar{v} = G_n(u, v, \bar{z}) = \bar{F}_n(v) + f_n(u, v, \bar{z}), \quad z = q_n(u, v, \bar{z}), \quad (45)$$

where p_n, q_n, f_n are smooth functions defined for small (u, \bar{z}) and for v from a small neighbourhood A'' of \bar{A} in A . We assume

$$f_n(0, v, 0) \equiv 0 \quad (46)$$

in (45). As $u = 0$ corresponds to the initial point in W_{loc}^u , and $\bar{z} = 0$ corresponds to the image of this point (by T_n) that lies in W_{loc}^s , having both $u = 0$ and $\bar{z} = 0$ corresponds to the initial point lying in B_n^- and its image lying in B_n^+ . Thus, by (44), we have

$$\bar{F}_n = \Phi^{m_++m_-}|_{B_n^-} = F_0^{m_+} \circ F_n \circ F_0^{m_-}, \quad (47)$$

where F_n is the scattering map. Since the cylinder \bar{A} is invariant with respect to F_0 , the maps \bar{F}_n are defined in a neighbourhood of \bar{A} , as the scattering maps F_n are. Thus, we will

further assume that the open neighbourhood A'' of \bar{A} in A is chosen such that the *modified scattering maps* \bar{F}_n are all defined there and are homotopic to identity diffeomorphisms of A'' into A , moreover

$$F_0^{-m-}(A') \subseteq A'' \quad (48)$$

where A' is a small neighbourhood of \bar{A} in A where the scattering maps F_n are defined.

We recall that we assume that the invariant cylinder A persists for the class of perturbations of the map Φ we consider here (the perturbations are the maps $\tilde{\Phi}$ from a small neighbourhood \mathcal{V} of Φ in the set of real-analytic exact symplectic maps), and A depends continuously on the map. Its stable and unstable manifolds, as well as the corresponding strong-stable and strong-unstable foliations, also depend on the map $\tilde{\Phi}$ continuously (in C^1). The transversality condition [S1] implies that the C^1 -smooth homoclinic cylinders B_1, \dots, B_N also persist and depend continuously on the map. Thus, for all maps from \mathcal{V} , the scattering maps F_1, \dots, F_N are defined on the same open subset A' of A and the modified scattering maps $\bar{F}_1, \dots, \bar{F}_N$ are defined on an open subset A'' of A such that (48) holds. All these maps are exact symplectomorphisms, homotopic to identity. The maps T_n , $n = 1, \dots, N$, remain well-defined for all $\tilde{\Phi}$ from \mathcal{V} , and formulas (45)-(47) hold. Estimates (13)-(24) also remain valid (as they are true for every map with a compact, symmetrically normally-hyperbolic invariant cylinder A). In what follows we will consider arbitrary maps from \mathcal{V} (e.g. we do not assume the invariance of the cylinder \bar{A} for all such maps; from now on it is just a cylinder bounded by two arbitrary, non-intersecting, essential curves $\gamma^\pm \subset A'$).

Let us denote as T_0 the map $\tilde{\Phi}$ restricted to the δ -neighbourhood of A . Let us call the union of the δ -neighbourhood of A with certain small neighbourhoods of the cylinders $\tilde{\Phi}(B^-), \dots, \tilde{\Phi}^{m_++m_- - 1}(B^-)$ a *homoclinic channel*. For every finite orbit in the homoclinic channel there is a uniquely defined sequence of points P_s ($s = 0, \dots, 2J+1$) of this orbit which lie in the δ -neighbourhood of A and satisfy $P_{2j+1} = T_0^{k_j} P_{2j}$ ($j = 0, \dots, J$) and $P_{2j} = T_{n_j} P_{2j-1}$ ($j = 1, \dots, J$), where n_j may take values from $1, \dots, N$ and $k_j \geq 0$. We will call the sequence P_j a *channel orbit*, and the sequence $k_0, \{n_s, k_s\}_{1 \leq s \leq J}$ will be called *the code* of the orbit. Given a code $k_0, \{n_s, k_s\}_{1 \leq s \leq J}$, we may consider a *shadow orbit*, namely a sequence v_s^* of points in A such that $v_{2j+1}^* = F_0^{k_j} v_{2j}^*$ and $v_{2j}^* = \bar{F}_{n_j} v_{2j-1}^*$. Note that we always assume that

$$v_{2j-1}^* \in A'', \quad j = 1, \dots, J, \quad (49)$$

in order to have the maps \bar{F}_{n_j} well-defined (this, in particular, implies that not all codes necessarily correspond to a shadow orbit). Given a channel orbit P_s , its shadow is the shadow orbit of the corresponding code with v_0^* equal to the v -coordinate of P_0 .

4.2 Shadowing orbits of proper codes

Our next goal is to estimate the deviation of the channel orbit P_s from its shadow. Here, we restrict our attention onto a special class of orbits which correspond to a special class of codes. Namely, we will call a finite code *proper* if the corresponding sequence k_s satisfies for every s

$$k_s \geq \bar{k} \quad \text{and} \quad k_s \geq \gamma k_{s+1} + D, \quad (50)$$

for some $\bar{k} \geq 0$, $D \geq 0$ and $\gamma > 1$. In other words, k_s is a sufficiently fast decreasing sequence of sufficiently large numbers.

Lemma 4.1 *Given any sufficiently large \bar{k} , γ and D , for any shadow orbit v_0^*, \dots, v_{2J+1}^* with a proper code $k_0, \{n_s, k_s\}_{1 \leq s \leq J}$, given any u^{in} and z^{out} such that $\|u^{in}\| \leq \delta$, $\|z^{out}\| \leq \delta$, in the δ -neighbourhood of A' there exists a uniquely defined channel orbit $P_s(u_s, v_s, z_s)$ such that $u_0 = u^{in}$, $v_0 = v_0^*$, $z_{2J+1} = z^{out}$, and $P_{2j+1} = T_0^{k_j} P_{2j}$, $P_{2j} = T_{n_j} P_{2j-1}$. Moreover,*

$$\|v_s - v_s^*\| \leq \delta(\alpha\lambda)^{k_J/2} \leq 2\delta(\alpha\lambda)^{\bar{k}/2}, \quad (51)$$

and

$$\|u_{2J+1}\| \leq \delta\lambda^{\bar{k}}, \quad \|z_0\| \leq \delta\lambda^{\bar{k}}. \quad (52)$$

Remark 4.2 Usual shadowing results would require hyperbolicity (or its topological analogues) from the maps F_0 and F_1, \dots, F_N , see e.g. [22]. We, however, do not make any assumption on the dynamics of these maps in this lemma (e.g. we have not assumed the symplecticity so far). Therefore we need to restrict here the class of shadow orbits to those with proper codes only; we believe any significantly stronger shadowing statement can not hold in this situation without further assumptions.

Proof of the lemma. For $J = 0$ the statement of the lemma is contained in Lemma 3.1, so we will proceed by induction in J . Thus, we may make an assumption that given any \tilde{z} such that $\|\tilde{z}\| \leq \delta$ the sequence (u_s, v_s, z_s) with the code k_0, n_1, \dots, k_{J-1} at $s = 0, \dots, 2J-1$ is uniquely defined by the condition $u_0 = u^{in}$, $v_0 = v_0^*$ and $z_{2J-1} = \tilde{z}$, and that the condition

$$\|v_s - v_s^*\| \leq \delta(\alpha\lambda)^{k_{J-1}/2} \quad (53)$$

is fulfilled for all $s \leq 2J-1$. Denote $u_{2J-1} = \tau(\tilde{z})$ and $v_{2J-1} = \phi(\tilde{z})$ (we assume u^{in} and v_0^* fixed, and do not indicate the dependence on them). Note that it follows from (49), (53) that

$$\phi(\tilde{z}) \in A''_\rho \quad \text{for any } \rho > \delta(\alpha\lambda)^{\bar{k}/2}, \quad (54)$$

where A''_ρ is the closed ρ -neighbourhood of A'' .

Since $(u_{2J-1}, v_{2J-1}, \tilde{z}) = T_0^{k_{J-1}}(u_{2J-2}, v_{2J-2}, z_{2J-2})$, it follows from Lemma 3.1 (see (20)) that

$$\|\tau\| \leq \delta\lambda^{k_{J-1}} \leq \delta\lambda^{\bar{k}}. \quad (55)$$

We will also include in our induction assumption a bound for the derivatives:

$$\|\tau', \phi'\| \leq \nu \quad (56)$$

for some sufficiently small constant ν . Thus, in order to carry out the induction, when we prove that the sought sequence (u_j, v_j, z_j) is uniquely defined for all $j = 0, \dots, 2J+1$ we must also show that

$$\|\partial(u_{2J+1}, v_{2J+1})/\partial z^{out}\| \leq \nu \quad (57)$$

with the same ν .

Since $(u_{2J+1}, v_{2J+1}, z^{out}) = T_0^{k_J}(u_{2J}, v_{2J}, z_{2J})$, it follows from Lemma 3.1 that z_{2J} is a uniquely defined smooth function of (u_{2J}, v_{2J}) and z^{out} ; we denote this function as $\sigma : (u_{2J}, v_{2J}, z^{out}) \mapsto z_{2J}$. By (20),(22),

$$\|\sigma\| \leq \delta \lambda^{k_J} \leq \delta \lambda^{\bar{k}}, \quad (58)$$

and

$$\|\sigma'\| \leq \nu, \quad (59)$$

for any ν chosen in advance (if \bar{k} is large enough), and

$$\|\partial\sigma/\partial z^{out}\| \leq \lambda^{k_J}. \quad (60)$$

With these notations, the sought sequence (u_j, v_j, z_j) exists indeed and is defined uniquely if and only if the following equation has a unique solution

$$\begin{aligned} u_{2J} &= p_{n_J}(\tau(z_{2J-1}), \phi(z_{2J-1}), \sigma(u_{2J}, v_{2J}, z^{out})), \\ v_{2J} &= G_{n_J}(\tau(z_{2J-1}), \phi(z_{2J-1}), \sigma(u_{2J}, v_{2J}, z^{out})), \\ z_{2J-1} &= q_{n_J}(\tau(z_{2J-1}), \phi(z_{2J-1}), \sigma(u_{2J}, v_{2J}, z^{out})), \end{aligned} \quad (61)$$

(see (45)). By construction, the functions p_n, q_n, G_n in (45) are defined at all u and \bar{z} which are sufficiently small and all $v \in A''$. Obviously, they will remain defined and smooth for v from a sufficiently small neighbourhood of A'' . Thus, to be sure that the system (61) is well-defined, we must check that, by taking \bar{k} sufficiently large, the values of τ and σ can be made arbitrarily small and the range of values of ϕ can be confined to an arbitrarily small neighbourhood of A'' , and this is indeed given by (54),(55),(58).

As the functions p_n, q_n, G_n are smooth, their derivatives are bounded by a constant. We, therefore, write

$$\|p'_n, q'_n, G'_n\| \leq C$$

and take ν in (56),(59) such that

$$C\nu < 1.$$

Thus, the right-hand side of system (61) is a contracting operator, which will immediately give us the existence and uniqueness of the solution once we check that given any z, u, v and z^{out} such that $\|z\| \leq \delta, \|z^{out}\| \leq \delta, \|u\| \leq \delta$ and $v \in A''_\rho$ (for some ρ small enough) the functions $p_n(\tau(z), \phi(z), \sigma(u, v, z^{out})), G_n(\tau(z), \phi(z), \sigma(u, v, z^{out})), q_n(\tau(z), \phi(z), \sigma(u, v, z^{out}))$ return the new values of, respectively, u, v and z for which the same conditions are satisfied (namely, $\|z\| \leq \delta, \|u\| \leq \delta$ and $v \in A''_\rho$). That $\|p_n\| \leq \delta$ and $\|q_n\| \leq \delta$, this follows immediately, as p_n and q_n define the map T_n which acts from a small neighbourhood of the cylinder B_n^- to a small neighbourhood of the cylinder B_n^+ , and both the cylinders lie inside the δ -neighbourhood of A . So, we are left to verify that $v_{2J} = G_{n_J}(\tau, \phi, \sigma) \in A''_\rho$. Note that

by the induction assumption (53) we have $\|\phi - v_{2J-1}^*\| \leq 2\delta(\alpha\lambda)^{k_{J-1}/2}$. Since $G_n := \bar{F}_n + f_n$, and $v_{2J}^* = \bar{F}_{n_J}(v_{2J-1}^*)$, it follows that

$$\|G_{n_J}(\tau, \phi, \sigma) - v_{2J}^*\| \leq C\|\phi - v_{2J-1}^*\| + \|f_{n_J}(\tau, \phi, \sigma)\|,$$

hence

$$\|v_{2J} - v_{2J}^*\| \leq C\delta(2(\alpha\lambda)^{k_{J-1}/2} + \lambda^{k_J}) \quad (62)$$

(we take into account that f_n vanishes, by (46), at $\tau = 0, \sigma = 0$, so $\|f_n\| \leq C\|\tau, \sigma\| \leq C\lambda^{k_J}$ by (55),(58)). At \bar{k} large enough, we have from (62) that

$$\|v_{2J} - v_{2J}^*\| \leq \rho.$$

Since v_{2J}^* lies in A'' , it follows that $v_{2J} = G_{n_J}(\tau, \phi, \sigma) \in A''_\rho$ indeed.

Thus, system (61) has a unique solution $(u_{2J}, v_{2J}, z_{2J-1})$ as required; moreover, by (60) we have

$$\|\partial(u_{2J}, v_{2J})/\partial z^{out}\| \leq \frac{C}{1 - C\nu}\lambda^{k_J} = o(\alpha^{-k_J}) \quad (63)$$

(as $\alpha\lambda < 1$ by (19)). Since $(u_{2J+1}, v_{2J+1}, z^{out}) = T_0^{k_J}(u_{2J}, v_{2J}, z_{2J})$, we find from (22),(23) and (63) that $\|\partial(u_{2J+1}, v_{2J+1})/\partial z^{out}\|$ can be made as small as we need by taking k_J large enough, i.e. (57) holds true indeed for \bar{k} large enough.

We have proved the existence and uniqueness of the sought sequence (u_s, v_s, z_s) , $s = 0, \dots, 2J+1$. It remains to prove inequality (51). At $j \leq 2J-1$ this is given by the induction assumption (53), since $k_{j-1} > k_j$. In order to check (51) at $j \geq 2J$, we recall that $(u_{2J+1}, v_{2J+1}, z_{2J+1}) = T_0^{k_J}(u_{2J}, v_{2J}, z_{2J})$ and $v_{2J+1}^* = F_0^{k_J}v_{2J}^*$. It then follows from (17), (62) and (21) that

$$\|v_{2J+1}^* - F_0^{k_J}v_{2J}\| \leq C\delta(2(\alpha\lambda)^{k_{J-1}/2} + \lambda^{k_J})\alpha^{k_J},$$

and

$$\|v_{2J+1} - F_0^{k_J}v_{2J}\| \leq \delta(\alpha\lambda)^{k_J/2},$$

which gives

$$\|v_{2J+1} - v_{2J+1}^*\| \leq \delta(\alpha\lambda)^{k_J/2}(1 + C(\alpha\lambda)^{\bar{k}/2}) + 2C\delta(\alpha\lambda)^{k_{J-1}/2}\alpha^{k_J}.$$

We finish the proof of the lemma by noting that this inequality and inequality (62) imply (51) at $j = 2J+1$ and, respectively, $j = 2J$ if

$$\gamma > \ln \frac{1}{\alpha\lambda} \Big/ \ln \frac{\alpha}{\lambda}, \quad D > 2 \ln \frac{2C}{1 - C(\alpha\lambda)^{\bar{k}/2}} \Big/ \ln \frac{\alpha}{\lambda}$$

in (50). Inequality (52) follows immediately from (20). \square

4.3 Replacing a code with a proper code

If the diffeomorphism F_0 is area-preserving, the Poincare Recurrence Theorem implies that recurrent (Poisson stable) orbits of F_0 are dense in the invariant cylinder A . This fact, as the following lemma shows, allows an arbitrary orbit of the iterated function system $\{F_0, F_1, \dots, F_N\}$ to be approximated by a shadow with a proper code.

Lemma 4.3 *Let v_0, \dots, v_{2J+1} be a sequence of points in $\text{int}(A)$ such that*

$$v_{2j+1} = F_0^{i_j}(v_{2j}) \quad (j = 0, \dots, J), \quad v_{2j} = F_{n_j}(v_{2j-1}) \quad (j = 1, \dots, J) \quad (64)$$

for some $i_j \geq 0$ and $n_j = 1, \dots, N$ (we assume that the points v_{2j-1} , $j = 1, \dots, J$, belong to the open set A' , the domain of definition of the maps F_n with $n \geq 1$). Then, given two open subsets of A , an arbitrary neighbourhood U_0 of the point v_0 and an arbitrary neighbourhood U_{2J+1} of the point v_{2J+1} , and given any \bar{k} , γ and D , in $\text{int}(A)$ there exists a sequence of points v_s^ such that*

$$v_0^* \in U_0, \quad v_{2J+1}^* \in U_{2J+1}, \quad v_{2j+1}^* = F_0^{k_j}(v_{2j}^*) \quad (j = 0, \dots, J), \quad v_{2j}^* = \bar{F}_{n_j}(v_{2j-1}^*) \quad (j = 1, \dots, J)$$

with the same n_j as in (64), $v_{2j-1}^ \in A''$ (the domain of definition of the maps \bar{F}_n) for $j = 1, \dots, J$, and the numbers k_j form a proper sequence in the sense of (50).*

Proof. By the definition of the modified scattering maps \bar{F}_n (see (47)), it is enough to show that in $\text{int}(A)$ there exists a sequence of points \hat{v}_s such that $\hat{v}_0 \in U_0$, $\hat{v}_{2J+1} \in U_{2J+1}$,

$$\hat{v}_{2j+1} = F_0^{\hat{k}_j}(\hat{v}_{2j}) \quad (j = 0, \dots, J), \quad \hat{v}_{2j} = F_{n_j}(\hat{v}_{2j-1}) \quad (j = 1, \dots, J) \quad (65)$$

with the same n_j as in (64), $\hat{v}_{2j-1} \in A'$ for $j = 1, \dots, J$, and the numbers \hat{k}_j are such that $\hat{k}_J = k_J + m_+$, and, if $J \geq 1$, then $\hat{k}_j = k_j + m_+ + m_-$ at $0 \leq j \leq J-1$, where k_j , $j = 0, \dots, J$, form a proper sequence. The sought sequence v_s^* will then be given by

$$v_0^* = \hat{v}_0, \quad v_{2J+1}^* = \hat{v}_{2J+1}, \quad v_{2j-1}^* = F_0^{-m_-} \hat{v}_{2j-1}, \quad v_{2j}^* = F_0^{m_+} \hat{v}_{2j} \quad (j = 1, \dots, J);$$

we will have $v_{2j-1}^* \in A''$ by virtue of (48).

We will proceed by induction in J . Let \hat{U} be a small neighbourhood of v_1 in A such that $F_0^{-i_0} \hat{U} \subset U_0$. If $J = 0$ we assume $\hat{U} \subseteq U_1$; if $J \geq 1$, we assume that \hat{U} is small enough, so that the map F_{n_1} is defined everywhere on \hat{U} . By induction, there is a point $v' \in U_2 = F_{n_1} \hat{U}$ such that $\mathcal{F}v' \in U_{2J+1}$, where $\mathcal{F} = \left(\prod_{2 \leq j \leq J} F_0^{\hat{k}_j} \circ F_{n_j} \right) \circ F_0^{\hat{k}_1}$ with the numbers \hat{k}_j such that, if we define

$$k_J = \hat{k}_J - m_+ \quad \text{and, at} \quad J \geq 2, \quad k_j = \hat{k}_j - m_- - m_+ \quad \text{for all} \quad j \leq J-1, \quad (66)$$

then the numbers k_j , $j = 1, \dots, J$, form a proper sequence. Let $U' \subseteq \hat{U}$ be a small enough neighbourhood of v_1 in A such that $\mathcal{F} \circ F_{n_1}(U') \subseteq U_{2J+1}$; at $J = 0$ take $U' = \hat{U} \subseteq U_1$.

By construction, we will prove the lemma if we find a point $\hat{v}_1 \in U'$ such that $\hat{v}_0 = F_0^{-\hat{k}_0} \hat{v}_1 \in U_0$ for some \hat{k}_0 such that

$$\hat{k}_0 - m_- - m_+ \geq \bar{k} \quad (67)$$

and, at $J \geq 1$,

$$\hat{k}_0 - m_- - m_+ \geq \gamma k_1 + D, \quad (68)$$

with k_1 given by (66). Indeed, if we define $k_0 = \hat{k}_0 - m_+$ at $J = 0$ and $k_0 = \hat{k}_0 - m_+ - m_-$ at $J \geq 1$, then the sequence k_0, \dots, k_J will be proper (see (50)), and for the corresponding sequence \hat{v}_s defined by (65) we will have $\hat{v}_{2J+1} \in U_{2J+1}$, as $\hat{v}_{2J+1} = \mathcal{F} \circ F_{n_1}(\hat{v}_1) \in \mathcal{F} \circ F_{n_1}(U') \subseteq U_{2J+1}$ at $J \geq 1$ and $\hat{v}_{2J+1} = \hat{v}_1 \in U' \subseteq U_1$ at $J = 0$.

Now note that since F_0 is an area-preserving map of a bounded region A , recurrent orbits are dense in A . In particular, there is a recurrent point in U' . It is easy to see that any such point can serve as the sought point \hat{v}_1 . Indeed, if $\hat{v}_1 \in U'$ is recurrent, then one can find an arbitrarily large k such that $F_0^{-k} \hat{v}_1 \in U'$. As $U' \subseteq \hat{U}$, it follows that $F_0^{-i_0-k} \hat{v}_1 \in U_0$. So, we put $\hat{k}_0 = k + i_0$, and conditions (67), (68) are fulfilled if the return time k was taken large enough. \square

We say that two points v_0 and v_m are connected by an orbit of the iterated function system $\{F_0, \dots, F_N\}$ if v_{2J+1} is an image of v_0 by a certain sequence of maps F_n . Obviously, this means that v_0 and v_m are the first and the last points in a sequence of points v_s constructed by the rule (64) with $m = 2J+1$. Since the corresponding sequence v_s^* constructed in Lemma 4.3 is a shadow of proper code, we may apply Lemma 4.1 to it. Thus, combining Lemmas 4.3 and 4.1, we obtain the following

Lemma 4.4 *Let the map F_0 be area-preserving. Let two points $v_0 \in A$ and $v_m \in A$ be connected by an orbit of the iterated function system $\{F_0, \dots, F_N\}$. The, for any $\varepsilon > 0$ the ε -neighbourhoods of v_0 and v_m in \mathbb{R}^{2d} are connected by an orbit of the map $\tilde{\Phi}$.*

5 Transport in the homoclinic channel in the symplectic case

5.1 Symplecticity of invariant manifolds and scattering maps

In this Section we continue the analysis of the behaviour in the homoclinic channel. Now we take into account the fact that the map $\tilde{\Phi}$ under consideration is exact symplectic. We start with establishing some useful geometric properties of the stable and unstable manifolds and the scattering maps. These properties are based on a symplectic orthogonality property, as given by the following

Proposition 5.1 *If A is a symmetrically normally-hyperbolic invariant manifold and $x \in A$, then $T_y W^s(A) \perp_\Omega T_y E^{ss}(x)$ for any $y \in E^{ss}(x)$ and $T_y E^{uu}(x) \perp_\Omega T_y W^u(A)$ for any $y \in E^{uu}(x)$.*

Proof. Let $y \in W^s(A)$. Take any $w \in T_y E^{ss}(x)$ and $u \in T_y W^s(A)$. Since the map $\tilde{\Phi}$ preserves the form Ω , we have for any $m \in \mathbb{N}$:

$$\Omega(w, u) = \Omega((\tilde{\Phi}')^m w, (\tilde{\Phi}')^m u) = (\alpha\lambda)^m \Omega(\alpha^{-m}(\tilde{\Phi}')^m w, \lambda^{-m}(\tilde{\Phi}')^m u) = O((\alpha\lambda)^m).$$

Taking the limit $m \rightarrow +\infty$, we find that $\Omega(w, u) = 0$, i.e. $u \perp_\Omega w$. Thus, we have proved $T_y E^{ss}(x) \perp_\Omega T_y W^s(A)$. In a similar way we conclude that $T_y E^{uu}(x) \perp_\Omega T_y W^u(A)$ for any $y \in W^u(A)$. \square

Note that we do not use that the manifold A is a two-dimensional cylinder embedded in \mathbb{R}^{2d} . In fact, this and other statements we prove here (Propositions 5.2-5.5) assume only that A is a submanifold of a smooth manifold M endowed with a closed non-degenerate symplectic form Ω , and that the diffeomorphism $\tilde{\Phi}$ preserves Ω .

Proposition 5.2 *The restriction of the symplectic form Ω to the symmetrically normally-hyperbolic invariant manifold A is non-degenerate.*

Proof. If the proposition is not true and the restriction of the symplectic form is degenerate, then there is a non-zero vector $w \in T_x(A)$ such that $w \perp_\Omega T_x(A)$. On the other hand $w \in T_x A = T_x W^s(A) \cap T_x W^u(A)$ implies that $w \perp_\Omega T_x E_x^{ss}$ and $w \perp_\Omega T_x E_x^{uu}$. The normal hyperbolicity assumptions imply that $TM = T_x E_x^{ss} \oplus T_x E_x^{uu} \oplus T_x A$ for any $x \in A$. Consequently, $w \perp_\Omega T_x M$, which contradicts to the non-degeneracy of Ω , and the proposition follows immediately. \square

We call a homoclinic intersection of $W^u(A)$ and $W^s(A)$ at a point y strongly transverse if E_y^{uu} is transverse to $W^s(A)$ and E_y^{ss} is transverse to $W^u(A)$ at the point y .

Proposition 5.3 *If $y \in E^{uu}(x_1) \cap E^{ss}(x_2)$ for some $x_1, x_2 \in A$ and $TM = T_y E^{uu}(x_1) \oplus T_y W^s(A)$ then $TM = T_y E^{ss}(x_2) \oplus T_y W^u(A)$ and, consequently, the homoclinic intersection at y is strongly transverse.*

The proof of this proposition is completely straightforward: it is sufficient to note that any vector from $T_y E^{ss}(x_2) \cap T_y W^u(A)$ is Ω -orthogonal to all vectors due to Proposition 5.1. The proposition implies that the strong transversality is equivalent to the transversality of the strong stable leaves to the unstable manifold (or the transversality of the strong unstable leaves to the stable manifold). This property reduces the number of conditions which are necessary to verify the strong transversality of a homoclinic intersection.

For every $y \in W^s(A)$ there is a unique $x \in A$ such that $y \in E^{ss}(x)$. We define the projection $\pi^s : W^s(A) \rightarrow A$ by setting $\pi^s(y) = x$. Let u_1, u_2, \dots be some coordinates on A defined in a small neighbourhood U of the point x . Define coordinates (u, v) in $(\pi^s)^{-1}(U)$ such that $v = 0$ corresponds to a point in A and $u = \text{const}$ corresponds to the strong stable leave $E_{(p,q,0)}^{ss}$. In these coordinates $\pi^s : (u, v) \mapsto (u, 0)$.

Since $T_y E^{ss}(x) \perp_\Omega T_y W^s(A)$, we see that in these coordinates $\Omega|_{W^s(A)} = \sum_{i,j} a_{ij}(u, v) du_i \wedge du_j$. On the other hand, the symplectic form is closed, i.e., $d\Omega = 0$. We have $d\Omega|_{W^s(A)} =$

$\sum_{i,j} \frac{\partial a_{ij}}{\partial v} dv \wedge du_i \wedge du_j$, which implies that the coefficients a_{ij} do not depend on v , and consequently $\Omega|_{W^s(A)} = \sum_{i,j} a_{ij}(u) du_i \wedge du_j$.

Let B be any section of $W^s(A)$ transverse to the strongly stable leaves. Then the restriction $\pi^s|_B : B \rightarrow A$ is a local diffeomorphism. Moreover, since the projection is the identity in the coordinates u , we find that $\pi^s|_B$ is a symplectomorphism, i.e. it transforms $\Omega|_B$ into $\Omega|_A$. In particular, $\Omega|_B$ is non-degenerate, i.e. B is a symplectic manifold.

Obviously, a similar statement is true for the stable manifolds replaced by the unstable ones: for any section B of $W^u(A)$ transverse to the strongly unstable leaves, the projection $\pi^u : B \rightarrow A$ by the strongly unstable leaves is locally a symplectomorphism. Thus, we obtain the following

Proposition 5.4 *If $y \in W^s(A) \cap W^u(A)$ is a strongly transverse homoclinic point and B is a sufficiently small neighbourhood of y inside $W^s(A) \cap W^u(A)$, then the scattering map $F_B = \pi^s|_B \circ (\pi^u|_B)^{-1} : B^u \rightarrow B^s$ is a symplectomorphism, where $B^{u,s} = \pi^{u,s}(B) \subset A$.*

We can define the scattering map F_B relative to any connected subset B of $W^s(A) \cap W^u(A)$ that consists of strongly transverse homoclinic points. When B is not a small neighbourhood of a single point, the scattering map F_B does not need to be single-valued nor injective (eventhough every branch of it is a local diffeomorphism). In the example of simple homoclinic cylinder we consider in this paper, the scattering map is, however, single-valued and injective (so it is a symplectic diffeomorphism for a large open subset A' of A).

Assume the symplectic form is exact: $\Omega = d\vartheta$, where ϑ is a differential 1-form (in the case of our interest, $M = \mathbb{R}^{2d}$, $\Omega = dp \wedge dq$, and $\vartheta = pdq$). The symplectic map $\tilde{\Phi}$ is exact when

$$\int_{\gamma} \vartheta = \int_{\tilde{\Phi}(\gamma)} \vartheta$$

for every smooth closed curve γ . Obviously, the exactness of $\tilde{\Phi}$ implies the exactness of the map $F_0 = \tilde{\Phi}|_A$.

Proposition 5.5 *Let $A' \subseteq A$ be a region such that the scattering map F_B is a diffeomorphism $A' \rightarrow F_B(A')$; moreover, for each point $x \in A'$ the corresponding leaves $E^{uu}(x)$ and $E^{ss}(F_B(x))$ intersect B exactly at one point. Then the restriction of F_B on A' is an exact map.*

Proof. Let us prove that the map $(\pi^u|_B)^{-1}$ is exact on A' . The proof of the exactness of the map $(\pi^s|_B)^{-1}$ on $F_B(A')$ is exactly the same, so the exactness of F_B will follow immediately. Take any smooth closed curve $\gamma \subset A'$. By assumption, for any $x \in \gamma$ there is a unique point $y(x) \in B$ such that $y \in E^{uu}(x)$, the union of the points $y(x)$ over all $x \in \gamma$ gives the curve $(\pi^u|_B)^{-1}\gamma = \tilde{\gamma} \subset B$. As the strongly unstable leaves are simply-connected (each is a diffeomorphic copy of \mathbb{R}^k where $2k = \dim(M) - \dim(A)$) and depend smoothly on the base point x , one can connect each point $x \in \gamma$ with the corresponding point $y(x) \in \tilde{\gamma}$ by a smooth

arc $\ell(x)$ that lies in $E^{uu}(x)$ so that the union of these arcs forms a smooth two-dimensional surface $S \subset W^u(A)$, an annulus bounded by γ and $\tilde{\gamma}$. By Stokes theorem,

$$\int_{\gamma} \vartheta - \int_{\tilde{\gamma}} \vartheta = \int_S \Omega.$$

At every point $y \in S$ the tangent plane contains the vector tangent to one of the curves $\ell(x)$ which lies in the $E^{uu}(x)$, so Ω vanishes on $T_y S$ by Proposition 5.1. Thus, $\int_S \Omega = 0$, which gives us the required identity $\int_{\gamma} \vartheta = \int_{\tilde{\gamma}} \vartheta$ for every smooth closed curve γ in A' . \square

Note that, surprisingly, the exactness of the scattering map in the statement above does not require the exactness of the map $\tilde{\Phi}$ itself.

5.2 Transport in an iterated functions system and obstruction curves

The symplecticity of the map $F_0 = \tilde{\Phi}|_A$ established in Proposition 5.2 means that this map is area-preserving (with the area of a domain obtained by integrating $\Omega|_A$ over this domain). Therefore, as shown in Section 4.3, for two open sets to be connected by an orbit from the homoclinic channel it is enough for these sets to be connected by the orbits of the iterated function system $\{F_0, F_1, \dots, F_N\}$. As we showed in Section 5.1 all these maps are exact symplectomorphisms. The diffeomorphism F_0 is defined everywhere on the cylinder A which is invariant with respect to F_0 , i.e. $F_0(A) = A$. The scattering maps F_n , $n = 1, \dots, N$, are defined on a subset A' of the cylinder A and, as follows from the simplicity assumptions [S1]-[S3], they are homotopic to identity diffeomorphisms $A' \rightarrow A$. The exact symplecticity of the maps F_n implies that the area between any curve γ and its image $F_n(\gamma)$ is zero. Hence, $F_n(\gamma) \cap \gamma \neq \emptyset$ for any essential curve $\gamma \subset A'$.

We assume that there exist coordinates $v = (y, \varphi)$ in A such that the map $F_0 : (y, \varphi) \mapsto (\bar{\varphi}, \bar{y})$ in these coordinates satisfies the *twist condition*, i.e.

$$\frac{\partial \bar{\varphi}}{\partial y} \neq 0$$

everywhere in this cylinder (we assume that $\varphi \in \mathbb{S}^1$ is the angular variable). Recall that according to the Birkhoff theory, any invariant essential curve of a twist map is a graph of a Lipschitz function $y(\varphi)$ [49].

Let \bar{A} be a compact cylinder in A' bounded by two simple essential curves γ^+ and γ^- such that $\gamma^- \cap \gamma^+ = \emptyset$ (we no longer need to assume that \bar{A} is an invariant cylinder). Let γ^+ corresponds to larger values of y than γ^- does. The set $A \setminus \text{int}(\bar{A})$ consists of two connected components, the upper component A^+ contains γ^+ and the lower component A^- contains γ^- . If \bar{A} contains an essential curve γ^* which is *invariant under all of the maps F_n , $n = 0, \dots, N$* , then the curve γ^* separates the cylinder \bar{A} into two invariant parts, so no trajectory of the iterated function system $\{F_0, F_1, \dots, F_N\}$ which starts in A^- can get to A^+ . In other words, the absence of essential common invariant curves in \bar{A} is a necessary condition for the orbits of iterated function system to connect A^- with A^+ . The following theorem shows that this condition is also sufficient.

Theorem 5.6 *Let F_1, \dots, F_N be exact symplectomorphisms $A' \rightarrow A$, homotopic to identity. Let A be invariant with respect to a symplectic diffeomorphism F_0 which satisfies the twist condition on A . Suppose no essential curve in \bar{A} is invariant with respect to all maps F_n , $n = 0, 1, \dots, N$. Then there is a finite trajectory $(v_i)_{i=0}^m \subset \bar{A}$ of the iterated function system $\{F_0, F_1, \dots, F_N\}$ that starts on γ^- and ends on γ^+ (i.e. $v_0 \in \gamma^-$, $v_m \in \gamma^+$, and $v_{i+1} = F_{k_i}(v_i)$ for some sequence of $k_i = 0, \dots, N$).*

Remark 5.7 As the common invariant curve is, in particular, an invariant curve of the twist map F_0 , the Birkhoff theory implies that it is necessarily must be a graph of a Lipschitz function $y = y^*(\varphi)$, so it is sufficient to verify the absence of Lipschitz common invariant curves.

Remark 5.8 Our statement makes an important change in the setup of the problem compared to e.g. [72, 14] as we do not ask the boundaries γ^- and γ^+ to be invariant with respect to any of the maps F_n , $n = 0, \dots, N$. Indeed, it is not natural to assume that the scattering maps preserve boundaries as this would require certain non-transversality of stable and unstable manifolds associated with $\tilde{\Phi}$ -invariant curves on the boundary.

Proof of Theorem 5.6. The boundary of the F_0 -invariant cylinder A consists of two non-intersecting essential curves; we will refer to the boundary curve with larger values of the coordinate y as the upper boundary of A . Let $\gamma \subset A'$ be a simple essential curve and let γ^n be the boundary of the connected component of $A \setminus (\gamma \cup F_n(\gamma))$ adjacent to the upper boundary of A . This is also a simple essential curve. Denote the operator that replaces the curve γ by γ^n as \mathcal{F}_n . By construction, $\mathcal{F}_n(\gamma)$ has no points below γ and, since the map F_n is exact, $\gamma \cap \mathcal{F}_n(\gamma) \neq \emptyset$. If $\mathcal{F}_n(\gamma) \cap \gamma^+ \neq \emptyset$ for some n , the existence of the sought connecting orbit is trivial (indeed, take $v_1 \in \mathcal{F}_n(\gamma) \cap \gamma^+$ and $v_0 = F_n^{-1}(v_1)$).

Now we continue by induction. Let $m = 0$, $\gamma_0 = \gamma^-$ and inductively construct a sequence of simple essential curves $\gamma_m \subset \bar{A}$, such that each point on γ_m can be reached by a trajectory which starts on γ^- and has the length not longer than m . Suppose we have constructed such γ_m for some $m \geq 0$. If $\mathcal{F}_n(\gamma_m) \cap \gamma^+ \neq \emptyset$ for some j , the inductive process is terminated as the intersection point belongs to the sought trajectory which starts on γ^- and finishes on γ^+ . Otherwise define γ_{m+1} as the boundary of that connected component of $A \setminus (\cup_n \mathcal{F}_n(\gamma_m))$ which is adjacent to the upper boundary of A . Obviously, γ^+ is a simple essential curve. Since each of the curves $\mathcal{F}_n(\gamma_m)$ intersects $\gamma_m \subset \bar{A}$ (by the exactness of F_n) and none of them, by assumption, intersects γ^+ , it follows that $\gamma_{m+1} \subset \bar{A}$. By construction, γ_{m+1} has no points below γ_m .

We claim that this process will terminate after a finite number of steps because otherwise the maps F_n would have a common invariant essential curve in \bar{A} . Indeed, suppose that the process does not terminate. Recall that the inductive process produces the sequence of curves which is “bounded and monotone” in the sense that γ_{m+1} lies in the region between γ_m and γ^+ for all m . Let U^* denote that connected component of $A \setminus (\cup_{m \geq 0} \gamma_m)$ which is adjacent to the upper boundary of A and let γ^* be the boundary of U^* .

Let us show that $F_0(U^*) = U^*$. Indeed, suppose this is not the case. Then we cannot have $U^* \subset F_0(U^*)$, as this would imply that the set $F_0(U^*) \setminus U^*$ has non-empty interior, hence the area of $F_0(U^*)$ will be strictly larger than the area of U^* which is impossible as F_0 is area-preserving. Thus, $U^* \not\subset F_0(U^*)$, which means $F_0^{-1}(U^*) \not\subset U^*$. At the same time $F_0^{-1}(U^*) \cap U^* \neq \emptyset$, so, since U^* is connected, $F_0^{-1}(U^*)$ intersects the boundary γ^* of U^* , i.e. there exists a point $p^* \in \gamma^*$ such that $F_0(p^*) \in U^*$. By the construction of γ^* , there is a sequence of points $p_m \in \gamma_m$ which converges to p^* . Consequently, since U^* is open, $F_0(p_m) \in U^*$ for all sufficiently large m . This is impossible as $F_0(p_m) \in F_0(\gamma_m)$, and the curve $F_0(\gamma_m)$ cannot have points above γ_{m+1} , i.e. it cannot intersect U^* .

Now note that since U^* is invariant with respect to the twist map F_0 and contains an essential invariant curve of F_0 (the upper boundary of A), the Birkhoff theory implies that the boundary γ^* of U^* is a simple essential curve, a graph of a Lipschitz function $y(\varphi)$. Let us show that this curve is a common invariant curve for all of the maps F_n . Indeed, let γ^* be not invariant with respect to F_n for some n . Then, since F_n is exact symplectic, $F_n(\gamma^*)$ must have points on both sides from γ^* , i.e. there exists $p^* \in \gamma^*$ such that $F_n(p^*) \in U^*$. As γ^* lies in the closure of the curves γ_m , there is a sequence of points $p_m \in \gamma_m$ which converges to p^* . Consequently, $F_n(p_m) \in U^*$ for all sufficiently large m , which is impossible as $F_n(p_m) \in F_n(\gamma_m)$, and the curve $F_n(\gamma_m)$ cannot have points in U^* .

We have proved that the non-existence of the trajectory of statement 1 is equivalent to the existence of a common invariant curve of statement 2. \square

Theorem 5.6 is valid for any two non-intersecting essential curves in A' : either they are connected by an orbit of the iterated function system, or there is an essential curve γ^* between them which is invariant with respect to all maps F_n . It follows that the absence of a common invariant essential curve in \bar{A} is equivalent to the existence of an orbit of the iterated function systems which connects $\text{int}(A^+)$ with $\text{int}(A^-)$ (move the curves γ^+ and γ^- inside $\text{int}(A^+)$ and, respectively, $\text{int}(A^-)$, and apply Theorem 5.6 to these curves). Since the existence of such orbit is an open property, Theorem 5.6 implies that the cylinder \bar{A} contains no essential curve invariant with respect to all maps F_0, \dots, F_N for an *open* set of maps from \mathcal{V} . In the next Section we show that this set of maps is also *dense* in \mathcal{V} . This will finish the proof of the Main Theorem: it follows immediately from Theorem 5.6 and Lemma 4.4 that for any map $\tilde{\Phi}$ from this open and dense set any two neighbourhoods of γ^- and γ^+ are connected by $\tilde{\Phi}$.

6 Simultaneous destruction of all obstruction curves

We finish the proof of the Main Theorem by showing that for a dense subset of the neighbourhood \mathcal{V} of our original map Φ the maps F_0, F_1, \dots, F_N do not have a common essential invariant curve. As we mentioned, it is enough for us to prove it for any given fixed N . As F_0 is a twist map, we can restrict the problem to Lipschitz invariant curves only. Recall that for any map $\tilde{\Phi}$ from \mathcal{V} there exists a compact normally-hyperbolic invariant cylinder A . We introduce coordinates (y, φ) on A such that the restriction F_0 of $\tilde{\Phi}$ on A has a twist property.

Namely, by denoting $F_0 : (y, \varphi) \mapsto (\bar{y}, \bar{\varphi})$ we have

$$\frac{\partial \bar{y}}{\partial \varphi} \neq 0$$

for all $(y, \varphi) \in A$. By a Birkhoff theorem, every essential invariant curve of F_0 is Lipschitz:

$$y = y(\varphi), \quad |y(\varphi_1) - y(\varphi_2)| \leq L|\varphi_1 - \varphi_2|,$$

where the Lipschitz constant L satisfies

$$L \leq \sup_{v \in \bar{A}} \max \left\{ \left| \frac{\partial \bar{\varphi}}{\partial \varphi} \right| / \left| \frac{\partial \bar{\varphi}}{\partial y} \right|, \left| \frac{\partial \bar{y}}{\partial y} \right| / \left| \frac{\partial \bar{\varphi}}{\partial y} \right| \right\}.$$

Since all the maps from \mathcal{V} are C^1 -close and the corresponding cylinders A are C^1 -close as well, we can take the constant L the same for all maps from \mathcal{V} .

By assumptions of the Main Theorem and by Proposition 3.3, we have a compact sub-cylinder \bar{A} in A such that $N \geq 8$ scattering maps are defined on a neighbourhood A' of \bar{A} . The cylinder \bar{A} depends continuously on the map $\tilde{\Phi}$, so we can choose A' to be the same (in appropriately chosen coordinates (y, φ)) for all maps close to $\tilde{\Phi}$. We can also assume that the maps F_1, \dots, F_N are defined in some neighbourhood of the closure of A' . Note that the scattering maps depend continuously on the map $\tilde{\Phi}$ in the following sense: if Φ is C^2 -close to $\tilde{\Phi}$, then the corresponding scattering maps are C^1 -close.

Theorem 6.1 *Arbitrarily close to any map $\tilde{\Phi}$ in \mathcal{V} there exists a map for which the corresponding scattering maps F_1, \dots, F_8 have no common L -Lipschitz invariant curves in A' .*

Proof. Consider the space of all L -Lipshitz functions $y = y(\varphi)$ endowed by the C^0 -metrics. Let \mathcal{L} be the subset of this space which consists of all functions whose graphs lie in the closure of A' and are invariant, simultaneously, for all the scattering maps F_1, \dots, F_8 generated by the map $\tilde{\Phi}$. If $\mathcal{L} = \emptyset$, there is nothing to prove. If $\mathcal{L} \neq \emptyset$, we note that \mathcal{L} is compact, so given any $\delta > 0$ there is a finite set of L -Lipshitz curves C_1, \dots, C_q such that each of them is invariant with respect to all the maps F_1, \dots, F_8 and every other common invariant L -Lipshitz curve lies in the δ -neighbourhood of one of the curves C_s , i.e. it belongs to the cylinder $A_s := \{|y - y_s(\varphi)| \leq \delta\}$ where $y = y_s(\varphi)$ is the equation of the curve C_s . Moreover, the set of the L -Lipshitz common invariant curves of the scattering maps depends upper-semicontinuously on the map $\tilde{\Phi}$ (if we have a sequence of maps $\Phi^{(k)}$ that converges to $\tilde{\Phi}$ in C^2 , then the corresponding scattering maps $F_j^{(k)}$ converge to the scattering maps F_j in C^1 ; and if the maps $F_j^{(k)}$ each have an L -Lipshitz invariant curve, then the set of the limit points of these curves as $k \rightarrow +\infty$ is the union of a set of L -Lipshitz curves each of which is invariant with respect to the scattering maps F_j). Thus, for all maps from \mathcal{V} which are sufficiently close to $\tilde{\Phi}$, every common invariant L -Lipshitz curve of the scattering maps that lies in A' lies entirely in one of the cylinders A_1, \dots, A_q .

Below (see (71)) we will fix, once and for all, a certain value of $\delta > 0$ which will give us a finite set of these cylinders A_s . We will show for each such cylinder A_s that arbitrarily close

to $\tilde{\Phi}$ in \mathcal{V} there exists a map for which the corresponding scattering maps F_1, \dots, F_8 have no common L -Lipshitz invariant curves in A_s . This will prove the theorem. Indeed, the absence of the common invariant L -Lipshitz curves in any given (open) cylinder is an open property. So, we first perturb the map $\tilde{\Phi}$ such that to kill all common invariant L -Lipshitz curves in the cylinder A_1 , then we add one more small perturbation to kill all common invariant L -Lipshitz curves in A_2 - by choosing the perturbation small enough we guarantee that no new common invariant L -Lipshitz curves emerge in A_1 , etc.. Then, after finitely many steps of the procedure we will have all the cylinders A_1, \dots, A_q cleaned of the possible common invariant L -Lipshitz curves.

Let $R > 1$ be a constant that bounds the derivatives of the scattering maps:

$$\left\| \frac{\partial F_j}{\partial(y, \varphi)} \right\| < R \quad (69)$$

for all $(y, \varphi) \in A'$, $j = 1, \dots, 8$, and all $\Phi \in \mathcal{V}$ that close enough to $\tilde{\Phi}$. Recall that φ is an angular variable that runs a circle \mathbb{S}^1 ; we assume that the length of the circle is 2π . Choose 4 arcs $J_i \subset \mathbb{S}^1$, $i \in \{1, 2, 3, 4\}$, $J_1 \cup J_2 = J_3 \cup J_4 = \mathbb{S}^1$. Moreover, denote $J_{ik} = J_i \setminus J_k$ and let us assume that J_{12} , J_{34} , J_{21} and J_{43} are disjoint and lie in \mathbb{S}^1 in the same order as they are listed here (following the orientation on the circle \mathbb{S}^1). Neither of the arcs J_i constitutes the whole circle, so their lengths are smaller than 2π . Choose any L -Lipshitz curve $C : y = y_C(\varphi)$ which is invariant with respect to all maps F_1, \dots, F_8 . Each arc J_i corresponds to an arc $\hat{J}_i : \{y = y_C(\varphi), \varphi \in J_i\}$ of the curve C . Since C is invariant with respect to each of the maps F_j , the image $F_j(\hat{J}_i)$ also lies in C . Hence it is given by $F_j(\hat{J}_i) : \{y = y_C(\varphi), \varphi \in \bar{J}_i^j\}$ where \bar{J}_i^j is a certain arc in \mathbb{S}^1 which does not cover the whole of \mathbb{S}^1 , so its length is strictly less than 2π . Since the set \mathcal{L} of all common invariant L -Lipshitz curves is compact, we have

$$K = \max_{C \in \mathcal{L}} \max_{i,j} \text{length}(\bar{J}_i^j) < 2\pi. \quad (70)$$

Now, we choose

$$\delta = \frac{2\pi - K}{R} > 0. \quad (71)$$

As it was explained above, the compactness of \mathcal{L} implies that every possible common invariant L -Lipshitz curve lies in one of a finitely many cylinders A_s ; each of these cylinders is the δ -neighbourhood of some invariant L -Lipshitz curve $C_s : \{y = y_s(\varphi)\}$. Take any of these cylinders. Note that, by virtue of (69), the image $F_j(A_s \cap \{\varphi \in J_i\})$ lies inside the $(R\delta)$ -neighbourhood of the curve $F_j(C_s \cap \{\varphi \in J_i\})$. This curve is a subset of the invariant curve C_s , and it corresponds to an interval of φ values such that the length of this interval does not exceed the constant K defined by (70). Thus, by (71),

$$F_j(A_s \cap \{\varphi \in J_i\}) \subset \{|y - y_s(\varphi)| < R\delta, \varphi \in \hat{J}_{sij}\} \quad (72)$$

where \hat{J}_{sij} is a certain arc whose length is strictly less than 2π , i.e. it does not cover the entire \mathbb{S}^1 . As F_j depends continuously on the map $\tilde{\Phi}$, inclusion (72) holds for all maps from \mathcal{V} which are close enough to $\tilde{\Phi}$.

Now, let us imbed the map $\tilde{\Phi}$ into a two-parameter analytic family of maps Φ_{μ_1, μ_2} from \mathcal{V} such that $\Phi_0 = \tilde{\Phi}$. We will show (Lemmas 6.2 and 6.3) that this family can be chosen such that there exist arbitrarily small values of $\mu = (\mu_1, \mu_2)$ for which the scattering maps F_1, \dots, F_8 defined by the map Φ_μ have no common L -Lipschitz invariant curves in the cylinder A_s . The map Φ_μ that corresponds to a small value of μ is a small perturbation of $\tilde{\Phi}$, so this gives us the required arbitrarily small perturbations that clear the cylinder A_s of the common L -Lipshitz invariant curves of the scattering maps. By performing this perturbations consecutively for each of the cylinders A_1, \dots, A_q we will obtain the result of the theorem.

Note that the invariant cylinder A , its stable and unstable manifolds, as well as the strong stable and strong unstable foliations depend smoothly on μ , therefore the scattering maps also depend smoothly on μ . Denote

$$F_j : (y, \varphi) \mapsto (Y_j(y, \varphi, \mu), \Psi_j(y, \varphi, \mu)).$$

Let our family Φ_μ be chosen such that for all $(\varphi, y) \in A_s$

$$\left\| \frac{\partial \Psi_j}{\partial (\mu_1, \mu_2)} \right\| < 1 \quad \text{for all } j = 1, \dots, 8, \quad (73)$$

$$\left| \frac{\partial Y_{1,2,3,4}}{\partial \mu_2} \right| < 1, \quad \left| \frac{\partial Y_{5,6,7,8}}{\partial \mu_1} \right| < 1, \quad (74)$$

$$j = 1, 2 : \quad \frac{\partial Y_j}{\partial \mu_1} > 2(L+1) \quad \text{and} \quad \frac{\partial Y_{j+4}}{\partial \mu_2} > 2(L+1) \quad \text{when } \Phi_j(\varphi, y, \mu) \in J_j, \quad (75)$$

$$j = 3, 4 : \quad \frac{\partial Y_j}{\partial \mu_1} < -2(L+1) \quad \text{and} \quad \frac{\partial Y_{j+4}}{\partial \mu_2} < -2(L+1) \quad \text{when } \Phi_j(\varphi, y, \mu) \in J_j, \quad (76)$$

where L is the Lipschitz constant in the condition of the theorem, and J_{1234} are the arcs defined above.

We will show below (Lemma 6.3) that such family Φ_μ exists. Thus, we will prove the theorem by showing the following result.

Lemma 6.2 *For every family of maps Φ_μ , $\mu = (\mu_1, \mu_2)$, such that the derivatives of the scattering maps F_1, \dots, F_8 satisfy estimates (73)-(76) for all $(\varphi, y) \in A_s$, the set of parameter values for which the scattering maps F_1, \dots, F_8 have an L -Lipshitz common invariant essential curve in A_s has measure zero. In particular, there exist arbitrarily small values of μ for which the maps F_1, \dots, F_8 have no L -Lipshitz common invariant essential curves in the cylinder A_s .*

Proof. Take any two, may be equal, values of μ : $\mu = \mu^*$ and $\mu = \mu^{**}$, such that at $\mu = \mu^*$ the maps F_1, \dots, F_8 have a common L -Lipschitz invariant curve $\mathcal{L}^* : \{y = y^*(\varphi), \varphi \in \mathbb{S}^1\} \subset A_s$ and at $\mu = \mu^{**}$ they have a common L -Lipschitz invariant curve $\mathcal{L}^{**} : \{y = y^{**}(\varphi), \varphi \in \mathbb{S}^1\} \subset A_s$. Let us show that the following condition holds:

$$\|\mu^* - \mu^{**}\| \leq R|y^*(0) - y^{**}(0)|, \quad (77)$$

where R is defined in (69) and $\|\mu\| = \max\{|\mu_1|, |\mu_2|\}$.

We note that without loosing in generality we may assume that

$$y^*(0) \geq y^{**}(0), \quad (78)$$

$$|\mu_2^* - \mu_2^{**}| \leq |\mu_1^* - \mu_1^{**}| \quad \text{and} \quad \mu_1^* \geq \mu_1^{**}. \quad (79)$$

If necessary, these inequalities can be achieved by swapping y and $(-y)$, μ and $(-\mu)$, $F_1 \leftrightarrow F_3$, $F_2 \leftrightarrow F_4$, $F_5 \leftrightarrow F_7$, $F_6 \leftrightarrow F_8$, as well as $\mu_1 \leftrightarrow \mu_2$ and $F_{1,2,3,4} \leftrightarrow F_{5,6,7,8}$. Conditions (73)-(76) are symmetric with respect to these changes.

Now suppose (77) is not true, i.e.

$$0 \leq y^*(0) - y^{**}(0) < \frac{\Delta\mu}{R}, \quad (80)$$

where

$$\Delta\mu = \mu_1^* - \mu_1^{**} > 0.$$

Since $J_3 \cup J_4 = \mathbb{S}^1$, we have that $\varphi = 0$ lies in least in one of the arcs J_3 or J_4 . For definiteness, we assume $0 \in J_3$. Let $(\varphi^*, \bar{y}^*) = F_3(0, y^*(0), \mu^*)$ and $(\varphi^{**}, \bar{y}^{**}) = F_3(0, y^{**}(0), \mu^{**})$, i.e.

$$\begin{aligned} \varphi^* &= \Psi_3(0, y^*(0), \mu^*), & \bar{y}^* &= Y_3(0, y^*(0), \mu^*), \\ \varphi^{**} &= \Psi_3(0, y^{**}(0), \mu^{**}), & \bar{y}^{**} &= Y_3(0, y^{**}(0), \mu^{**}). \end{aligned}$$

Standard estimates based on the mean value theorem and formulas (69),(73),(74),(76),(79),(80) imply that

$$|\varphi^{**} - \varphi^*| < 2\Delta\mu, \quad \bar{y}^* - \bar{y}^{**} < -2L\Delta\mu.$$

Since the curves $y = y^*(\varphi)$ and $y = y^{**}(\varphi)$ are invariant with respect to F_3 (at $\mu = \mu^*$ and $\mu = \mu^{**}$ respectively), it follows that $\bar{y}^* = y^*(\varphi^*)$, $\bar{y}^{**} = y^{**}(\varphi^{**})$. Because of the L -Lipschitz property, we find that

$$y^*(\varphi^*) - y^{**}(\varphi^*) = \bar{y}^* - \bar{y}^{**} + y^{**}(\varphi^{**}) - y^{**}(\varphi^*) < -2L\Delta\mu + 2L\Delta\mu < 0.$$

Then taking into account (78) we conclude that

$$\mathcal{L}^* \cap \mathcal{L}^{**} \neq \emptyset.$$

Now, let us call an arc $I \subset \mathbb{S}^1$ of the φ values *positive* if $y^*(\varphi) > y^{**}(\varphi)$ for all $\varphi \in \text{int}(I)$, and $y^*(\varphi) = y^{**}(\varphi)$ at the end points of I . We call an arc *negative*, if $y^*(\varphi) = y^{**}(\varphi)$ at its end points and $y^*(\varphi) < y^{**}(\varphi)$ on its interior. It is convenient to allow arcs to have empty interiors, i.e. any point from $\mathcal{L}^* \cap \mathcal{L}^{**}$ is considered to be both a positive and a negative arc at the same time.

We have just proved that there is at least one negative and at least one positive arc. Let I be a positive arc. Let $\mathcal{L}_I^* = \{y = y^*(\varphi), \varphi \in I\}$ and $\mathcal{L}_I^{**} = \{y = y^{**}(\varphi), \varphi \in I\}$ be the corresponding pieces of the curves \mathcal{L}^* and \mathcal{L}^{**} , and let $\mathcal{D}_I = \{y^*(\varphi) \geq y \geq y^{**}(\varphi), \varphi \in I\}$ be the region bounded by \mathcal{L}_I^* and \mathcal{L}_I^{**} . Let us show that if $I \subseteq J_j$ for $j = 1$ or $j = 2$, then, with

this j , the image of \mathcal{L}_I^* by the map F_j at $\mu = \mu^*$ lies strictly inside $\mathcal{L}_{I'}^*$, which corresponds to a positive arc I' and

$$\text{length}(I') > \Delta\mu > 0, \quad (81)$$

$$\text{area}(\mathcal{D}_{I'}) > \text{area}(\mathcal{D}_I). \quad (82)$$

Indeed, denote as F_j^* the map F_j at $\mu = \mu^*$ and F_j^{**} the map F_j at $\mu = \mu^{**}$. Take any point $M(\varphi, y^*(\varphi)) \in \mathcal{L}_I^*$, so $\varphi \in I$. Let $M^*(\varphi^*, y^*(\varphi^*)) \in \mathcal{L}^*$ be the image of M by the map F_j^* , and $M'(\varphi', y') \in F_j^{**}(\mathcal{L}_I^*)$ be the image of M by the map F_j^{**} . Since I is a positive arc, we have that for any $\varphi \in I$ the point M is either on the curve \mathcal{L}^{**} or above it. Since \mathcal{L}^{**} is invariant with respect to F_j^{**} , the point $M' = (\varphi', y')$ also does not lie below \mathcal{L}^{**} , i.e.

$$y' \geq y^{**}(\varphi'). \quad (83)$$

We have

$$\begin{aligned} \varphi' &= \Psi_j(\varphi, y^*(\varphi), \mu^{**}), & y' &= Y_j(\varphi, y^*(\varphi), \mu^{**}), \\ \varphi^* &= \Psi_j(\varphi, y^*(\varphi), \mu^*), & y^*(\varphi^*) &= Y_j(\varphi, y^*(\varphi), \mu^*). \end{aligned}$$

Thus, inequalities (73)-(75) imply that

$$|\varphi^* - \varphi'| < \Delta\mu, \quad y^*(\varphi^*) - y' > (2L + 1)\Delta\mu$$

(recall that we assume $I \subseteq J_j$, hence $\varphi \in J_j$). By (83) and the L -Lipschitz property of \mathcal{L}^{**} we obtain

$$y^*(\varphi^*) - y^{**}(\varphi^*) > (L + 1)\Delta\mu > 0, \quad (84)$$

and

$$y^*(\varphi') - y' > (L + 1)\Delta\mu > 0. \quad (85)$$

Denote $\tilde{F}_j(\varphi) = \Psi_j(\varphi, y^*(\varphi), \mu^*)$, i.e., \tilde{F}_j is the restriction of the map F_j^* on the invariant curve \mathcal{L}^* . We have just showed that if $\varphi \in I$, where $I \subseteq J_j$ is a positive arc, then $\varphi^* = \tilde{F}_j(\varphi)$ satisfies (84), i.e. it is inside some positive arc I' . Moreover, at the end points of I' we must have $y^* - y^{**} = 0$ while at the points of $\tilde{F}_j(I) \subseteq I'$ we have $y^* - y^{**} > L\Delta\mu$ by (84), hence the length of I' is bounded from below as in (81), by virtue of the $2L$ -Lipschitz property of the function $y^*(\varphi) - y^{**}(\varphi)$.

We have shown that $F_j^*(\mathcal{L}_I^*) \subset \mathcal{L}_{I'}^*$ and $F_j^{**}(\mathcal{L}_I^{**}) \subset \mathcal{L}_{I'}^{**}$ where I' is a positive arc. As the point M runs \mathcal{L}_I^* , the point $M'(\varphi', y')$ runs the curve $\mathcal{L}' = F_j^{**}(\mathcal{L}_I^*)$, and it follows from (83), (85) that the curve \mathcal{L}' lies between \mathcal{L}^* and \mathcal{L}^{**} , strictly below \mathcal{L}^* . Since the end points of \mathcal{L}' coincide with the end points of $F_j^{**}(\mathcal{L}_I^{**})$ and the latter lie inside $\mathcal{L}_{I'}^{**}$, it follows that \mathcal{L}' lies between $\mathcal{L}_{I'}^*$ and $\mathcal{L}_{I'}^{**}$, strictly below $\mathcal{L}_{I'}^*$. Therefore the area of the region $F_j^{**}(\mathcal{D}_I)$ bounded by the curves \mathcal{L}' and $F_j^{**}(\mathcal{L}_I^{**})$ is strictly smaller than the area of the region $\mathcal{D}_{I'}$ bounded by the curves $\mathcal{L}_{I'}^*$ and $\mathcal{L}_{I'}^{**}$. As the map F_j is area-preserving, $\text{area}(F_j^{**}\mathcal{D}_I) = \text{area}(\mathcal{D}_I)$, and (82) follows.

Thus, we start with any positive arc I which is contained entirely inside J_1 or J_2 and obtain a sequence I_s of positive arcs such that $I_0 = I$ and $\tilde{F}_{j_s}(I_s) \subset I_{s+1}$, where we chose $j_s = 1$ if $I_s \subseteq J_1$, and $j_s = 2$ if $I_s \subseteq J_2$ and $I_s \not\subseteq J_1$. If for some s the arc I_s is not entirely

contained neither in J_1 nor in J_2 , the sequence is terminated. By (82), the area of the region \mathcal{D}_{I_s} is a strictly increasing function of s , so the arcs with different s can never coincide. The definition of a positive arc implies that the intersection of interiors for two different positive arcs is always empty. Thus, the arcs $\text{int}(I_s)$ are mutually disjoint. By (81), no more than $\frac{2\pi}{\Delta\mu}$ of such arcs can coexist in \mathbb{S}^1 . We conclude that the sequence I_s must terminate. This means the last arc in the sequence is not contained entirely neither in J_1 nor in J_2 , i.e. we have proved that there is a positive arc I^+ such that both $I^+ \cap J_{12} \neq \emptyset$ and $I^+ \cap J_{21} \neq \emptyset$.

Similarly, one proves that there exists a negative arc I^- such that $I^- \cap J_{34} \neq \emptyset$ and $I^- \cap J_{43} \neq \emptyset$. Since J_{12} , J_{34} , J_{21} and J_{43} are placed on \mathbb{S}^1 in this order, we find that the interiors of I^+ and I^- intersect, which is impossible by the definition of positive and negative arcs. Thus, by contradiction, we have established estimate (77).

Let $\mathcal{M} \subset \mathbb{R}^2$ be the set of all μ such that the maps F_1, \dots, F_8 have at least one common L -Lipschitz invariant curve in the cylinder A_s . Let \mathcal{Y} be the set which consists of all intersection points of these curves with the axis $\varphi = 0$. By (77), for each $y_0 \in \mathcal{Y}$ there is exactly one $\mu \in \mathcal{M}$ such that the corresponding system of scattering maps has a common L -Lipschitz invariant curve that lies in A_s and intersects the line $\varphi = 0$ at $y = y_0$. Estimate (77) also implies that $y_0 \mapsto \mu$ is an R -Lipschitz function $\mathcal{Y} \rightarrow \mathcal{M}$. For any Lipschitz function from a subset of \mathbb{R} to \mathbb{R}^2 , the Lebesgue measure of the image vanishes. Thus, as \mathcal{Y} is a subset of an interval, it follows that $\text{mes}(\mathcal{M}) = 0$, which gives the lemma. \square

We stress that Lemma 6.2 holds for any family of symplectic maps Φ_μ for which conditions (73)-(76) are satisfied. In order to finish the prove of the theorem, it remains to show that such family can be taken inside the space \mathcal{V} of analytic exact-symplectic maps. This is given by the lemma below.

Lemma 6.3 *Any map $\tilde{\Phi} \in \mathcal{V}$ can be imbedded into an analytic family of analytic exact-symplectic maps Φ_μ that satisfies conditions (73)-(76).*

Proof. We define $\Phi_\mu = X_\mu \circ \tilde{\Phi}$, where X_μ is an analytic family of exact-symplectic maps such that $X_0 = \text{id}$. Specifically, we set $X_\mu u = X_{\mu_1}^{(1)} \circ X_{\mu_2}^{(2)} u$ where $X_{\mu_i}^{(i)}$ is the time- μ_i shift by the orbits of the vector field defined by an analytic Hamiltonian function H_i ($i = 1, 2$). Since we are interested in small μ , it is enough to check the fulfillment of conditions (73)-(76) at $\mu = 0$ only. Thus the family $\Phi_\mu = X_\mu \circ \tilde{\Phi}$ will satisfy (73)-(76) for all small μ , provided the conditions

$$\begin{aligned} & \left| \frac{\partial \Psi_j}{\partial \mu_i} \right|_{\mu_i=0} < 1 \quad (j = 1, \dots, 8), \\ \text{for all } \varphi \in \mathbb{S}^1 : & \left| \frac{\partial Y_j}{\partial \mu_i} \right|_{\mu_i=0} < 1 \quad (j = 9 - 4i, \dots, 12 - 4i), \end{aligned} \tag{86}$$

$$\begin{aligned}
& \text{for all } \varphi \in J_j \text{ with } j = 1, 2: \quad \left. \frac{\partial Y_{j+4(i-1)}}{\partial \mu_i} \right|_{\mu_i=0} > 2(L+1), \\
& \text{for all } \varphi \in J_j \text{ with } j = 3, 4: \quad \left. \frac{\partial Y_{j+4(i-1)}}{\partial \mu_i} \right|_{\mu_i=0} < -2(L+1),
\end{aligned} \tag{87}$$

are satisfied by the scattering maps for the families $\Phi_{\mu_i}^{(i)} = X_{\mu_i}^{(i)} \circ \tilde{\Phi}$, $i = 1, 2$, for all $(\varphi, y) \in A_s$.

Let us build a family of maps $X_{\mu_1}^{(1)}$ for which these conditions are satisfied (the construction for $i = 2$ is the same). Conditions (86),(87) are strict and involve only the first derivatives of the scattering maps. A C^2 -small change of the family $\Phi_{\mu_1}^{(1)}$ leads to a C^1 -small change of the strong-stable and strong-unstable foliations and, therefore, a C^1 -small change of the scattering maps. Thus, it is enough to build a C^2 -smooth family of maps $X_{\mu_1}^{(1)}$ (generated by a C^3 -smooth Hamiltonian $H^{(1)}$) such that the corresponding scattering maps satisfy (86)-(87). Then for any sufficiently C^3 -close approximation of $H^{(1)}$ by an analytic Hamiltonian (the analyticity of $H^{(1)}$ and $H^{(2)}$ is needed for the family Φ_μ to be analytic, i.e. lie in \mathcal{V}) conditions (73)-(76) will still be satisfied.

We will make the C^3 -smooth Hamiltonian $H^{(1)}$ localised in a small neighbourhood of the cylinders $\tilde{\Phi}(B_1)$, $\tilde{\Phi}(B_2)$, $\tilde{\Phi}(B_3)$, $\tilde{\Phi}(B_4)$. Thus, the maps $X_{\mu_1}^{(1)}$ will differ from identity only in a small neighbourhood of these cylinders, so the maps $\Phi_{\mu_1}^{(1)}$ will differ from $\tilde{\Phi}$ in a small neighbourhood of the cylinders B_1, \dots, B_4 only. The perturbation we build near one of these cylinders does not affect the scattering maps near the other cylinders, so we restrict our attention to the cylinder B_1 only. We further omit the index "1" whenever possible and denote $\tau = \mu_1$. Thus we consider a homoclinic cylinder B and continue with building a C^3 -smooth Hamiltonian H localised in a small neighbourhood of the cylinder $\tilde{\Phi}(B)$ such that for the corresponding flow map X_τ the derivative with respect to τ of the scattering map F defined by the map $\Phi_\tau = X_\tau \circ \tilde{\Phi}$ satisfies, for all $(\varphi, y) \in A_s$, the following inequalities:

$$\begin{aligned}
& \left| \frac{\partial \Psi}{\partial \tau} \right|_{\tau=0} < 1 \quad \text{for all } \varphi \in \mathbb{S}^1, \\
& \left. \frac{\partial Y}{\partial \tau} \right|_{\tau=0} > 2(L+1) \quad \text{for all } \varphi \in J,
\end{aligned} \tag{88}$$

where J is a certain arc that does not contain the whole \mathbb{S}^1 , and

$$F(A_s \cap \{\varphi \in J\}) \subset \{\varphi \in \hat{J}\} \tag{89}$$

where \hat{J} is an arc that does not contain the whole of \mathbb{S}^1 (see (72)).

Let w^u denote a piece of the unstable manifold $W^u(A)$ that contains the cylinder B (i.e. w^u is a small neighbourhood of the cylinder B in $W^u(A)$) and w^s be a small neighbourhood of $\tilde{\Phi}(B)$ in $W^s(A)$, so $B = \tilde{\Phi}(w^u) \cap w^s$. Since the map Φ_τ differs from Φ in a small neighbourhood of the cylinder B only, the pieces w^u and w^s do not depend on τ , nor the strong unstable foliation of the piece of $W^u(A)$ between A and w^u depends on τ , neither the strong stable foliation of the piece of $W^s(A)$ between w^s and A does. Thus, given any

C^1 -family of cylinders B_τ close to B the projection map $\pi_{B_\tau}^u : B_\tau \rightarrow A$ by the leaves of the strong unstable foliation is of class C^1 ; moreover, if two such families of cylinders are C^1 -close, then the corresponding projection maps $\pi_{B_\tau}^u$ are also C^1 -close. The same holds true for the projection map $\pi_{B'_\tau}^s : B'_\tau \rightarrow A$ by the leaves of the strong stable foliation, where we denote as B'_τ any C^1 -family of cylinders close to $\tilde{\Phi}(B)$. As the perturbation X_τ is localised in a small neighbourhood of the cylinder $\tilde{\Phi}(B)$, we find that the scattering map F satisfies

$$F = F_0^{-1} \circ \pi_{B'_\tau}^s \circ X_\tau \circ \tilde{\Phi} \circ (\pi_{B_\tau}^u)^{-1}, \quad (90)$$

where $B_\tau = w^u \cap \Phi_\tau^{-1}(w^s)$ is a homoclinic cylinder close to B , and $B'_\tau = \Phi_\tau(B_\tau)$. If we add to the family X_τ any C^1 -small perturbation localised in a small neighbourhood of $\tilde{\Phi}(B)$, this will result in C^1 -small perturbations of the family of cylinders B'_τ and B_τ . Thus, the perturbation to the corresponding family of scattering maps defined by (90) will be also C^1 -small. It follows that it is enough to build a C^1 -family of maps X_τ (generated by a C^2 -smooth Hamiltonian H) localised in a small neighbourhood of the cylinder $\tilde{\Phi}(B)$ such that the corresponding family of scattering maps satisfies (88). Then any C^3 -Hamiltonian which is C^2 -close to H and is localised in a small neighbourhood of $\tilde{\Phi}(B)$ will produce a family of scattering maps that still satisfies (88).

This reduction of smoothness requirement (from $H \in C^3$ to $H \in C^2$) is important since it allows to construct the Hamiltonian H such that the vector field it generates is tangent to the given homoclinic cylinder B (for which only C^2 -smoothness can be guaranteed by our spectral gap assumptions). Once this is done, the cylinder $\tilde{\Phi}(B)$ will be invariant with respect to the map X_τ , i.e. $\Phi_\tau(B) = \tilde{\Phi}(B)$ for all τ . This means the trajectory of B remains the same for all τ , i.e. it remains a homoclinic cylinder. Thus, formula (90) for the scattering map will recast as

$$F = F_0^{-1} \circ \pi_{\tilde{\Phi}(B)}^s \circ X_\tau \circ \tilde{\Phi} \circ (\pi_B^u)^{-1}, \quad (91)$$

and the only τ -dependent term in the right-hand side is X_τ .

In order to build the required Hamiltonian, we introduce C^2 -coordinates (x, v) near $\tilde{\Phi}(B)$ such that the cylinder $\tilde{\Phi}(B)$ is given by $x = 0$ (so v gives the coordinates on the cylinder and x runs a neighbourhood of zero in \mathbb{R}^{2d-2}). The cylinder is transverse to the strong-stable and strong-unstable foliations, so if we denote as $N(v)$ the direct sum of the tangents to the leaves of the strong-stable and unstable foliations that pass through the point $(x = 0, v) \in \tilde{\Phi}(B)$, then the field $N(v)$ will have a form $dv = P(v)dx$. Note that N depends smoothly on v (as the fields of tangents to the strong stable and strong unstable leaves are smooth when the large spectral gap assumption (6) is fulfilled), i.e. the function $P(v)$ is at least C^1 . As the homoclinic cylinder $\tilde{\Phi}(B)$ belongs both to the stable and unstable manifolds of A , it follows from Proposition 5.1 that a vector is tangent to $\tilde{\Phi}(B)$ if and only if it is Ω -orthogonal to N . Thus, the vector field $\tilde{X} = \Omega^{-1}\nabla H$ generated by the Hamiltonian H will be tangent to $\tilde{\Phi}(B)$ if the gradient of H is orthogonal to N at the points of $\tilde{\Phi}(B)$, i.e.

$$\frac{\partial H}{\partial x}(0, v) + \frac{\partial H}{\partial v}(0, v)P(v) = 0. \quad (92)$$

This condition is satisfied e.g. by any function of the form

$$H(x, v) = h(v) - \sum_{i=1}^{2d-2} x_i \int p_i(v_1 + s_1 x_i, v_2 + s_2 x_i) \xi(s_1, s_2) d^2 s$$

where h is any C^2 -function on $\tilde{\Phi}(B)$, the vector-function $p(v) = (p_1(v), \dots, p_{2d-2}(v))$ is given by $p(v) = h'(v)P(v)$, the x_i 's are the coordinates of the vector x , and $(v_1, v_2) = v$, and ξ is a C^2 -smooth function on a plane, localised in a small neighbourhood of zero, such that $\int \xi(s) d^2 s = 1$. Integrating by parts, we find

$$\frac{\partial H}{\partial x_i} = \int p_i(v + s x_i) [s \xi'(s) + \xi(s)] d^2 s, \quad \frac{\partial H}{\partial v_j} = \frac{\partial h}{\partial v_j}(v) + \sum_{i=1}^{2d-2} \int p_i(v + s x_i) \frac{\partial x_i}{\partial s_j} d^2 s.$$

By plugging $x = 0$ in these formulas, we see that (92) is satisfied indeed. Since $q \in C^1$ and $\xi \in C^2$, it follows that $H \in C^2$, so given any C^2 -function h on the cylinder $\tilde{\Phi}(B)$ we can extend it to a C^2 -function H defined in a neighbourhood of this cylinder, such that the vector field generated by the Hamiltonian H will be tangent to the cylinder.

As we explained above, under this condition the scattering map is given by (91), so the vector field

$$\tilde{F} = \left(\tilde{\Psi} = \frac{\partial \Psi}{\partial \tau} \Big|_{\tau=0}, \tilde{Y} = \frac{\partial Y}{\partial \tau} \Big|_{\tau=0} \right)$$

of the τ -derivatives of the scattering map F on the cylinder A is given by

$$\tilde{F} = \frac{\partial}{\partial v} \left(F_0^{-1} \circ \pi_{\tilde{\Phi}(B)}^s \right) \circ \tilde{X} \circ \tilde{\Phi} \circ (\pi_B^u)^{-1}, \quad (93)$$

where $\tilde{X} = \Omega^{-1}(v)h'(v)$ is the vector field of the flow on the cylinder $\tilde{\Phi}(B)$, which is generated by the Hamiltonian h ; we denote as $\Omega(v)$ the antisymmetric (2×2) -matrix that defines the restriction of the symplectic form on the cylinder at the point v . In order to satisfy (88), we need to have

$$|\tilde{\Psi}| < 1 \quad \text{for all } \varphi \in \mathbb{S}^1, \quad (94)$$

$$\tilde{Y} > 2(L+1) \quad \text{for all } \varphi \in J.$$

It is seen from (93) that if conditions (94) are satisfied by \tilde{F} for some choice of the vector field \tilde{X} , they will be satisfied by \tilde{F} for any C^0 -small perturbation of \tilde{X} . Thus, it is enough to find any C^1 -smooth Hamiltonian function $h(v)$ such that the field \tilde{F} defined by (93) satisfies (94), then for any C^2 -smooth function which is C^1 -close to h the derivative of the scattering map F with respect to τ will satisfy (88), and the lemma will be proven.

In order to build the sought C^1 -function $h(v)$, we introduce C^1 -coordinates $v = (\varphi, y)$ on the cylinder $\tilde{\Phi}(B)$ such that the diffeomorphism $F_0^{-1} \circ \pi_{\tilde{\Phi}(B)}^s : \tilde{\Phi}(B) \rightarrow A$ is identity. Then (93) recasts as

$$\tilde{F} = \tilde{X} \circ F|_{\tau=0}$$

(see (91)). As \tilde{X} is a Hamiltonian vector field, its φ -component is given by $-\omega^{-1}\frac{\partial h}{\partial y}$ and the y -component is $\omega^{-1}\frac{\partial h}{\partial \varphi}$, where the C^0 -function $\omega(\varphi, y) > 0$ is such that $\omega(\varphi, y) dy \wedge d\varphi$ is the symplectic form on the cylinder $\tilde{\Phi}(B)$. Thus, conditions (94) take the form

$$\left| \frac{\partial h}{\partial y} \right| < \omega(\varphi, y) \quad \text{for all } (\varphi, y) \in F(A_s),$$

$$\frac{\partial h}{\partial \varphi} > 2(L+1)\omega(\varphi, y) \quad \text{for all } (\varphi, y) \in F(A_s \cap \{\varphi \in J\}).$$

We finish the proof of the lemma and the theorem by noticing that these conditions are satisfied by a y -independent function h such that

$$h(\varphi) = M\varphi \text{ at } \varphi \in \hat{J}$$

where the constant M is given by $M = 1 + 2(L+1) \sup_{F(A_s)} \omega$, and the arc \hat{J} is defined by (89); since h must be periodic in φ , it is important that \hat{J} does not cover the whole of \mathbb{S}^1 . \square

References

- [1] V.I. Arnold, *Instability of dynamical systems with many degrees of freedom*. (Russian) Dokl. Akad. Nauk SSSR 156 1964 9–12.
- [2] Arnold, V.I.; Kozlov, V.V.; Neishtadt, A.I.; *Mathematical aspects of classical and celestial mechanics. Dynamical systems III*. Third edition. Encyclopaedia of Mathematical Sciences, 3. Springer-Verlag, Berlin, 2006.
- [3] P. Bernard, *Perturbation of a partially hyperbolic Hamiltonian system* (French). C. R. Acad. Sci. Paris Sr. I Math. 323 (1996), no. 2, 189–194.
- [4] P. Bernard, *Arnold’ diffusion: from the a priori unstable to the a priori stable case*. Proceedings of the International Congress of Mathematicians, Hyderabad, India, 2010. Volume III, Hindustan Book Agency, New Delhi, 2010, pp.1680–1700
- [5] Bernard, P. Large normally hyperbolic cylinders in a priori stable Hamiltonian systems. Ann. Henri Poincaré 11 (2010), no. 5, 929–942.
- [6] M. Berti, L. Biasco, P. Bolle, *Drift in phase space: a new variational mechanism with optimal diffusion time*. J. Math. Pures Appl. (9) 82 (2003), no. 6, 613–664
- [7] M. Berti, P. Bolle, *Fast Arnold diffusion in systems with three time scales*. Discrete Contin. Dyn. Syst. 8 (2002), no. 3, 795–811.
- [8] U. Bessi, *Arnold’s diffusion with two resonances*. J. Differential Equations 137 (1997), no. 2, 211–239
- [9] S. Bolotin, D. Treschev, *Unbounded growth of energy in nonautonomous Hamiltonian systems*. Nonlinearity 12 (1999), no. 2, 365–388
- [10] J. Bourgain, V. Kaloshin, *On diffusion in high-dimensional Hamiltonian systems*. J. Funct. Anal. 229 (2005), no. 1, 1–61
- [11] A. Bounemoura, E. Pennamen, *Instability for a priori unstable Hamiltonian systems: a dynamical approach*. Discrete Contin. Dyn. Syst. 32 (2012), no. 3, 753–793.
- [12] A. Bounemoura, J.-P. Marco, *Improved exponential stability for near-integrable quasi-convex Hamiltonians*. Nonlinearity 24 (2011), no. 1, 97–112
- [13] H.W. Broer, F.M. Tangerman, *From a differentiable to a real analytic perturbation theory, applications to the Kupka-Smale theorems*, Ergodic Theory and Dyn. Sys. 6 (1986) 345–362.
- [14] P. Le Calvez, *Drift for families of twist maps on the annulus*, Ergodic Theory and Dynamical Systems, vol. 27 (2007) 869–879

- [15] L. Chierchia, G. Gallavotti, *Drift and diffusion in phase space*. Ann. Inst. H. Poincaré Phys. Théor. 60 (1994), no. 1, 144 pp. (erratum, Ann. Inst. H. Poincaré Phys. Théor. 68 (1998), no. 1, 135)
- [16] Chong-Qing Cheng, Jun Yan, *Existence of diffusion orbits in a priori unstable Hamiltonian systems*. J. Differential Geom. 67 (2004), no. 3, 457–517
- [17] Chong-Qing Cheng, Jun Yan, *Arnold diffusion in Hamiltonian systems: a priori unstable case*. J. Differential Geom. 82 (2009), no. 2, 229–277.
- [18] Chong-Qing Cheng, *Arnold diffusion in nearly integrable Hamiltonian systems* Preprint arXiv:1207.4016v2 (2013) 127 pp.
- [19] J. Cresson, *Symbolic dynamics and Arnold diffusion*. J. Differential Equations 187 (2003), no. 2, 269–292.
- [20] J. Cresson, S. Wiggins, *A λ -lemma for normally-hyperbolic invariant manifolds*, preprint (2005).
- [21] A. Delshams, V. Gelfreich, A. Jorba, T.-M. Seara, *Exponentially small splitting of separatrices under fast quasiperiodic forcing*, Communications in mathematical physics 189 (1997), 35–71.
- [22] A. Delshams, M. Gidea, P. Roldán, *Transition map and shadowing lemma for normally hyperbolic invariant manifolds*. Discrete Contin. Dyn. Syst. 33 (2013), no. 3, 1089–1112.
- [23] A. Delshams, G. Huguet, *Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems*. Nonlinearity 22 (2009), no. 8, 1997–2077.
- [24] A. Delshams, G. Huguet, *A geometric mechanism of diffusion: rigorous verification in a priori unstable Hamiltonian systems*. J. Differential Equations 250 (2011), no. 5, 2601–2623.
- [25] A. Delshams, R. de la Llave, T.M. Seara, *A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of \mathbf{T}^2* . Comm. Math. Phys. 209 (2000), no. 2, 353–392
- [26] A. Delshams, R. de la Llave, T.M. Seara, *Orbits of unbounded energy in quasi-periodic perturbations of geodesic flows*. Adv. Math. 202 (2006), no. 1, 64–188
- [27] A. Delshams, R. de la Llave, T.M. Seara, *A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model*. Mem. Amer. Math. Soc. 179 (2006), no. 844, viii+141 pp.
- [28] A. Delshams, R. de la Llave, T.M. Seara, *Geometric properties of the scattering map of a normally hyperbolic invariant manifold*. Adv. Math. 217 (2008), no. 3, 1096–1153

- [29] A. Delshams, R. de la Llave, T.M. Seara, *Geometric approaches to the problem of instability in Hamiltonian systems. An informal presentation* in W.Chreg (ed.) *Hamiltonian Dynamical Systems and Applications*, Springer (2008) pp.285–336
- [30] A. Delshams, R. de la Llave, T.M. Seara, *Instability of high dimensional Hamiltonian Systems: Multiple resonances do not impede diffusion*. (751K, pdf) Jun 17, 2013. preprint on mp_arc 13–56
- [31] R. Douady, *Stabilité ou instabilité des points fixes elliptiques* (French) [Stability or instability of elliptic fixed points] *Ann. Sci. école Norm. Sup.* (4) 21 (1988), no. 1, 1–46.
- [32] R. Douady, P. Le Calvez, *Example of a non-topologically stable elliptic fixed point in dimension 4* (French). *C. R. Acad. Sci. Paris Sér. I Math.* 296 (1983), no. 21, 895–898.
- [33] R. W. Easton, J.D. Meiss, G. Roberts, *Drift by coupling to an anti-integrable limit*. *Phys. D* 156 (2001), no. 3–4, 201–218.
- [34] E. Fermi. *Beweis, dass ein mechanisches Normalsystem im allgemeinen quasi-ergodisch ist*. *Physikalische Zeitschrift*, 24(??):261265
- [35] E. Fontich, P. Martin, *Arnold diffusion in perturbations of analytic exact symplectic maps*. *Nonlinear Anal.* 42 (2000), no. 8, 1397–1412
- [36] E. Fontich, P. Martin, *Arnold diffusion in perturbations of analytic integrable Hamiltonian systems*. *Discrete Contin. Dynam. Systems* 7 (2001), no. 1, 61–84.
- [37] G. Gallavotti, *Arnold’s diffusion in isochronous systems*. *Math. Phys. Anal. Geom.* 1 (1998/99), no. 4, 295–312
- [38] G. Gallavotti, G. Gentile, V. Mastropietro, *Hamilton-Jacobi equation, heteroclinic chains and Arnold diffusion in three time scale systems*. *Nonlinearity* 13 (2000), no. 2, 323–34
- [39] V. Gelfreich, D. Turaev, *Unbounded Energy Growth in Hamiltonian Systems with a Slowly Varying Parameter*, *Comm. Math. Phys.* vol. 283, no. 3 (2008) 769–794
- [40] V. Gelfreich, D. Turaev, *Fermi acceleration in non-autonomous billiards*, *J. Phys. A* 41, 212003 (2008)
- [41] M. Gidea, C. Robinson, *Shadowing orbits for transition chains of invariant tori alternating with Birkhoff zones of instability*. *Nonlinearity* 20 (2007), no. 5, 1115–1143
- [42] M. Gidea, C. Robinson, *Obstruction argument for transition chains of tori interspersed with gaps*. *Discrete Contin. Dyn. Syst. Ser. S* 2 (2009), no. 2, 393–416.
- [43] M. Gidea, P. Zgliczynski, *Covering relations for multidimensional dynamical systems. II*. *J. Differential Equations* 202 (2004), no. 1, 59–80

- [44] M. Gidea, R. de la Llave, *Topological methods in the instability problem of Hamiltonian systems*. Discrete Contin. Dyn. Syst. 14 (2006), no. 2, 295–328.
- [45] M. Gidea, R. de la Llave, T. Seara, *A general mechanism of diffusion in Hamiltonian systems: qualitative results*, preprint
- [46] M. Guzzo, E. Lega, C. Froeschlé, *First numerical evidence of global Arnold diffusion in quasi-integrable systems*. Discrete Contin. Dyn. Syst. Ser. B 5 (2005), no. 3, 687–698
- [47] M. Guzzo, E. Lega, C. Froeschlé, *A numerical study of the topology of normally hyperbolic invariant manifolds supporting Arnold diffusion in quasi-integrable systems*. Phys. D 238 (2009), no. 17, 1797–1807
- [48] M. Guzzo, E. Lega, C. Froeschlé, *A numerical study of Arnold diffusion in a priori unstable systems*. Comm. Math. Phys. 290 (2009), no. 2, 557–576.
- [49] M.-R. Herman, *Sur les courbes invariantes par les difféomorphismes de l’anneau*. Vol. 1. (French) [On the curves invariant under diffeomorphisms of the annulus. Vol. 1] Astérisque, 103-104. Soc. Math. de France, Paris, 1983. i+221 pp.
- [50] M.W. Hirsch, C.C. Pugh, M. Shub, *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977. 149 pp.
- [51] P.J. Holmes, J.E. Marsden, *Melnikov’s method and Arnold diffusion for perturbations of integrable Hamiltonian systems*. J. Math. Phys. 23 (1982), no. 4, 669–675
- [52] C.K.R.T. Jones, Siu-Kei Tin, *Generalized exchange lemmas and orbits heteroclinic to invariant manifolds*. Discrete Contin. Dyn. Syst. Ser. S 2 (2009), no. 4, 967–1023.
- [53] V. Yu. Kaloshin *Some prevalent properties of smooth dynamical systems*, Proc. Steklov Inst. Math. 2 (1996) 115–140.
- [54] V. Kaloshin, J. Mather, E. Valdinoci, *Instability of resonant totally elliptic points of symplectic maps in dimension 4*. Analyse complexe, systemes dynamiques, sommabilité des séries divergentes et théories galoisiennes. II. Astérisque No. 297 (2004), 79–116.
- [55] V. Kaloshin, *Geometric proofs of Mather’s connecting and accelerating theorems*. Topics in dynamics and ergodic theory, 81–106, London Math. Soc. Lecture Note Ser., 310, Cambridge Univ. Press, Cambridge, 2003
- [56] V. Kaloshin, M. Levi, *Geometry of Arnold diffusion*. SIAM Rev. 50 (2008), no. 4, 702–720
- [57] V. Kaloshin, M. Levi, *An example of Arnold diffusion for near-integrable Hamiltonians*. Bull. Amer. Math. Soc. (N.S.) 45 (2008), no. 3, 409–427

- [58] V. Kaloshin, M. Saprykina, *An example of a nearly integrable Hamiltonian system with a trajectory dense in a set of maximal Hausdorff dimension*. Comm. Math. Phys. 315 (2012), no. 3, 643–697.
- [59] V. Kaloshin, Ke Zhang, *A strong form of Arnold diffusion for two and a half degrees of freedom*, preprint arXiv:1212.1150 (2012) 207pp
- [60] V. Kaloshin and K. Zhang, *A strong form of Arnold diffusion for three and a half degrees of freedom*, preprint.
- [61] E. Lega, M. Guzzo, C. Froeschlé, *A numerical study of the size of the homoclinic tangle of hyperbolic tori and its correlation with Arnold diffusion in Hamiltonian systems*. Celestial Mech. Dynam. Astronom. 107 (2010), no. 1-2, 129–144
- [62] A. Litvak-Hinenzon, V. Rom-Kedar, *On energy surfaces and the resonance web*. SIAM J. Appl. Dyn. Syst. 3 (2004), no. 4, 525–573
- [63] R. de la Llave, *Some recent progress in geometric methods in the instability problem in Hamiltonian mechanics*. International Congress of Mathematicians. Vol. II, 1705–1729, Eur. Math. Soc., Zrich, 2006.
- [64] P. Lochak, J.-P. Marco, *Diffusion times and stability exponents for nearly integrable analytic systems*. Cent. Eur. J. Math. 3 (2005), no. 3, 342–397
- [65] P. Lochak, J.-P. Marco, D. Sauzin, *On the splitting of invariant manifolds in multi-dimensional near-integrable Hamiltonian systems*. Mem. Amer. Math. Soc. 163 (2003), no. 775, viii+145 pp.
- [66] J.-P. Marco, *Transition along chains of invariant tori for analytic Hamiltonian systems* (French). Ann. Inst. H. Poincaré Phys. Théor. 64 (1996), no. 2, 205–252.
- [67] J.-P. Marco, D. Sauzin, *Wandering domains and random walks in Gevrey near-integrable systems*. Ergodic Theory Dynam. Systems 24 (2004), no. 5, 1619–1666
- [68] J.-P. Marco, preprint
- [69] L. Markus, K.R. Meyer, *Generic Hamiltonian dynamical systems are neither integrable nor ergodic*. Memoirs of the AMS **144** (1974) 52 pp.
- [70] J.N. Mather, *Arnold diffusion: announcement of results*, J. Math. Sci. (N. Y.) 124 (2004), no. 5, 5275–5289
- [71] J.N. Mather, *Arnold diffusion by variational methods*. Essays in mathematics and its applications, 271–285, Springer, Heidelberg, 2012
- [72] R. Moeckel, *Generic drift on Cantor sets of annuli*. Celestial mechanics (Evanston, IL, 1999), 163–171, Contemp. Math., 292, Amer. Math. Soc., Providence, RI, 2002.

- [73] R. Moeckel, *Transition tori in the five-body problem*. J. Differential Equations 129 (1996).
- [74] M. Nassiri, E.R. Pujals, *Robust transitivity in Hamiltonian dynamics*. Ann. Sci. c. Norm. Supr. (4) 45 (2012), no. 2, 191–239.
- [75] N. N. Nekhoroshev, *An exponential estimate of the time of stability of nearly integrable Hamiltonian systems*. Russian Math. Surveys 32 (1977), no. 6, 1–65.
- [76] J. Palis, W. de Melo, *A geometrical introduction to dynamical systems*, Springer, 1982.
- [77] G.N. Piftankin, D.V. Treshchev, *Separatrix maps in Hamiltonian systems*. Russian Math. Surveys 62 (2007), no. 2, 219–322
- [78] M. Procesi, *Exponentially small splitting and Arnold diffusion for multiple time scale systems*. Rev. Math. Phys. 15 (2003), no. 4, 339–386
- [79] C. Pugh, M. Shub, A. Wilkinson, *Hölder foliations revisited*, Preprint (2011) 52 p.
- [80] C.R. Robinson, *Generic properties of conservative systems*, American Journal of Mathematics, vol. 92, no. 3 (1970) 562–603
- [81] C. Robinson, *Symbolic dynamics for transition tori*. Celestial mechanics (Evanston, IL, 1999), 199–208, Contemp. Math., 292, Amer. Math. Soc., Providence, RI, 2002.
- [82] L.P. Shilnikov, *On a Poincare-Birkhoff problem*, Mat. Sb. (1967).
- [83] L.P. Shilnikov, A. Shilnikov, D. Turaev, L. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics. Part I*. World Sci. 1998
- [84] D.V. Treschev, *Evolution of slow variables in a priori unstable Hamiltonian systems*. Nonlinearity 17 (2004), no. 5, 1803–1841
- [85] D. Treschev, *Arnold diffusion far from strong resonances in multidimensional a priori unstable Hamiltonian systems*. Nonlinearity 25 (2012), no. 9, 2717–2757.
- [86] J.L. Tennyson, M.A. Lieberman, A.J. Lichtenberg, *Diffusion in near-integrable Hamiltonian systems with three degrees of freedom*, in: M. Month, J.C. Herrera (Eds.), *Nonlinear Dynamics and the BeamBeam Interaction*, Vol. 57, American Institute of Physics, New York, 1979, pp. 272–301.
- [87] K. Tsiganis, A. Anastasiadis, H. Varvoglis, *Dimensionality differences between sticky and non-sticky chaotic trajectory segments in a 3D Hamiltonian system*, Chaos, Solitons and Fractals 11 (2000) 2281–2292